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Teachers College, Associate Dean Caswell stated at the William David Reeve Testimonial Dinner, conducted by the students of the mathematics department at Teachers College during the past summer, "Teachers College has been fortunate indeed in having had Professor Reeve's services for these many years. Professor Reeve has been a vigorous leader in improvement of curricula and instruction in mathematics. His influence has been nation-wide. During his period of service, some four thousand teachers of mathematics from all parts of the country have received inspiration, insight and guidance from him for the important work in which they were engaged. Under the direction of Professor Reeve and other members of the department, some sixty-five persons have completed the work for the doctorate. The high esteem in which he is held by his present and former students is evidenced by their generous gift to the college, establishing the William David Reeve Scholarship Fund. It gives me great personal pleasure to congratulate Professor Reeve upon his contribution to the College and to extend to him best wishes for continued years of service and happiness."

In their designating this number of THE MATHEMATICS TEACHER as the "Reeve Number," the Directors of the National Council are saying "Thank you, Reeve, for all that you have contributed and are contributing to our common cause. We recognize and appreciate your invaluable service."

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Mathematics and the Core Curriculum

By HAROLD P. FAWCETT
Ohio State University, Columbus, Ohio

WITHIN recent years many critical questions have been raised concerning the effectiveness with which the schools of America are meeting the needs of a growing and developing democracy. Significant social and economic changes have tended to focus attention on institutions and practices created to serve the demands of an earlier day. A greatly increased school population has emphasized the importance of general education and new theories of learning have raised serious doubt about the value of educational procedures designed to serve an intellectual aristocracy. Many an influential voice has asked penetrating questions concerning the adequacy of a curriculum based on the fundamental assumption that "knowledge is power" and increasing attention has been given to the development of those characteristics of personality considered essential to responsible citizenship in a democratic society.

Against this background of confusion and uncertainty earnest and thoughtful people, believing that the schools of America should be great laboratories of democracy, have proposed a fundamental curricular reorganization designed to yield this desirable result. They recognize the important fact that democratic outcomes can only be achieved by means that are also democratic. People tend to become what they do. Dictatorial methods will not yield democratic behavior. Critical thinking cannot be developed in an atmosphere which stifles critical thinking. Respect for the rights of others can be "learned" most effectively through methods that call for behavior consistent with that outcome nor is knowledge a one-way road that leads to power. Knowledge may be a necessary condition to power but it is not a sufficient condition and the intellectual highways of the world are crowded with weary wanderers burdened with an excess of non-functional knowledge, and yearning for the power that knowledge alone can never yield.

As a possible means of improving this situation "the core curriculum" is claiming the increasing attention of thoughtful men and women who are interested in developing an educational program appropriate to the needs of our democratic culture. What is the nature of the core curriculum? Approximately 300 years ago it meant the subject matter required of all students and it is used in precisely the same sense by the twelve distinguished men who wrote "General Education in a Free Society."[1] In early colonial days when every township of one hundred householders was required to establish a secondary school it meant only Latin and Greek while in the Harvard report it means eight units of secondary school work—"three in English, three in science and mathematics and two in the social studies."[2]

During the three centuries that have elapsed since the days of the Latin grammar school the core curriculum has steadily evolved from the subjects required of all to a unitary group of activities "planned by each pupil in his emerging experiences in cooperation with other pupils and under advice of members of the teaching staff."[3] It is not surprising that an evolving concept such as "the core" has no fixed or precise meaning but it is hardly possible to consider the relation of mathematics to this kind of a program unless we know something of its characteristics. In a thorough and comprehensive treatment of

[1] Harvard Committee, *General Education in a Free Society*, Cambridge, Massachusetts, Harvard University Press, 1945.

[2] *Ibid.*, p. 100.

[3] Hopkins, L. Thomas, *Integration—Its Meaning and Application*, New York, D. Appleton Century Company, 1937, pp. 234–252.

this important curricular development Professor Alberty[4] defines the core "as that aspect of the total curriculum which is basic for all students and which consists of learning activities that are organized without reference to conventional subject lines." He then proceeds to list eleven characteristics of core teaching—two of which are found in all current practices. These two are:

1. The core consists of learning activities that are regarded as basic to the education of all students. Within this framework, however, provision is made for individual differences.
2. The learning activities cut across conventional subject matter lines. This may involve "putting two or more subjects together" or complete disregard of boundaries.

From these considerations it is evident that "the core" still refers to those "common learnings" required of all students but the modern concept of what these common learnings are is vastly different from that of 1647. The emphasis today is on those learnings considered appropriate for effective participation in a democratic culture and knowledge becomes important to the extent that it is functional in improving the quality and character of life. An increasing body of opinion supports the position that these highly desirable results can best be achieved through the core curriculum and the vigorous voices of many students unite in support of this position. Through their school experiences they have learned to define problems, to gather relevant data, to formulate hypotheses and to test their validity. They know what it means to make use of a wide variety of resources in their study of a given problem and they recognize the limitations of a single reference. They have learned to work effectively with others for they had experience in planning with their classmates concerning problems of common interest and they know what it means to respect the opinion of their associates. An evaluation of the core curriculum, however, as interesting as that might be, is not within the scope of this discussion. The problem rather is to consider the relation of mathematics to this kind of a program and that is a problem which should claim the attention of all thoughtful teachers in this important field of learning. There is little question concerning the potentialities of the core for the development of social learnings appropriate to the needs of an expanding democracy. What kind of a contribution is this type of program prepared to make to the teaching and learning of mathematics?

One of the questions frequently asked concerning any core organization as earlier defined is whether or not mathematics is "in the core." That question was heard frequently during the Eight-Year-Study[5] which provided inviting opportunities for curricular experimentation and it is heard today. The variety of answers given to this interesting question have been highly suggestive but satisfying to no one. Of course mathematics is "in the corre" for if the core is regarded as "that aspect of of the total curriculum which is basic for all students" it will, of necessity, include mathematics. Is there any school anywhere operating on the theory that the curricular experiences "basic for all students" should not include mathematics? Is anyone prepared to support the proposition that a man who is mathematically illiterate is appropriately educated for responsible citizenship in this atomic age? Any adequate study of "the major problems of living" calls for the use of mathematical concepts, principles and processes and to discuss the question as to whether or not mathematics is "in the core" or "out of the core" is comparable to discussing the question as to whether or not language as a means of communication is "in the core" or "out of the core." Mathematics is not an inert body of knowledge to be moved

[4] Alberty, Harold, *Reorganization of the High School Curriculum*, New York, The Macmillan Company, pp. 154–155, 1947.

[5] Aikin, Wilford, *The Story of the Eight-Year-Study*, New York, Harper and Brothers, 1941.

'in" and "out" of a core at the whim of any given teacher or any group of teachers. It provides a language and a method which are essential to an intelligent study of any core problem. How, for example, can the students of Santa Barbara County, California, study the problem of "Spending Your Money Wisely"[6] without using the methods, ideas and techniques of mathematics? Does not the problem of "Preventing Accidents"[7] call for the securing of data concerning the number of accidents, the construction and interpretation of tables and graphs and the drawing of valid conclusions? Would any thoughtful teacher study the problem of "Providing Public Education"[8] and not consider such relevant data as the number of children to be served, the size of the plant needed, the cost of school buildings, salaries of teachers, taxes and the like? There is no question as to whether or not mathematics is "in the core." The real question is the effectiveness with which mathematics can be learned through core procedures and that is a question which commands our careful consideration.

One of the characteristics common to most all cores is a large block of time, usually from two to three hours, which provides flexibility in the planning of activities suited to the needs of the situation. It is within this block of time that learning activities "basic for all students . . . are organized without reference to conventional subject lines." Should mathematics be "in the core" in the sense that it is to be taught within this block of time? Is mathematics best learned withing such a framework? Fortunately we do not have to depend on theory and unsupported opinion as we seek to answer these important questions for this problem has already been studied by interested people and their findings cannot be ignored.

One such study entitled "Opportunities for the Use of Arithmetic in an Activity Program" is reported by Paul Hanna.[9] Six teachers of Grade 3 and six teachers of Grade 6 were selected to assist in the study and in the seven schools represented by these twelve teachers "a strictly separate subject approach with a systematic course of study" was not followed. The children's experiences were not organized "with reference to conventional subject lines" but rather were organized and integrated around worthwhile interests.[10] "Each teacher was asked to record on prepared blanks every situation faced by individuals or by her entire class, in which there was a need for quantitative thinking and manipulation" and in this connection it should be carefully noted that the problems recorded were not to originate in a source external to the unit being studied but were to be "those problems which arose in the pursuit of some child-selected activity.[11] These records were kept carefully over a period of four months and in the six third grades the total number of such quantitative problems which were *essential* "in the pursuit of some child-selected activity" was 234. Only 56% of these, however, called for actual computation which means that, on the average, only five computational problems per room per month were included. The "child-selected activity" in the six sixth grades yielded a total of 205 quantitative problems in 72% of which actual computation was needed. This means that if the arithmetic program of these children were limited to the quantitative work essential to the study of the core problem there would have been on the average approximately six computational problems per month for each of the sixth grade rooms. In view of these findings, it is not surprising that included in

[6] Selected from problems suggested in the *Santa Barbara County Curriculum Guide for Teachers in Secondary Schools*, 1941.

[7] *Ibid.*

[8] *Ibid.*

[9] Hanna, Paul, *Opportunities for the Use of Arithmetic in an Activity Program*, Tenth Yearbook, National Council of Teachers of Mathematics, 1935, pp. 85–120.

[10] *Ibid.*, p. 90.

[11] *Ibid.*

the report are the following statements.[12]

1. Functional experiences of childhood are alone not adequate to develop arithmetic skills.
2. The teacher should regard these meaningful arithmetic experiences as readiness . . . and must provide sufficient periods of practice to assure mastery.
3. Some guide or criterion in addition to opportunities-found-in-activities is essential to the selection of arithmetic materials.

The report also emphasizes the fact that "there is a large element of chance operating in the selection of units of work" and since the mathematics involved is determined by the type of unit selected it seems fair to say that any such educational procedure leaves to chance the mathematics studied.

Another interesting and significant study related to this problem was made by Catharine Williams.[13] While it is limited to one teacher and one grade in a single school the data were gathered over a nine year period and although "a single year was selected for the base year" the conclusions are derived from the evidence of nine successive years. The concept of the "experience curriculum" held by Miss Williams is also very significant. While it does not include "the cramping restrictions or expectations of predetermined organized subject matter requirements" neither is is it to be "limited to a concept of curriculum based entirely on direct first-hand contacts with reality. It is not to be confused with a more or less casual opportunistic dependence on incidental learning" and the pupils "are guided into a realization of the ways in which meaningful learnings can be broadened, extended and applied."[14] Consistent with this philosophy Miss Williams skillfully uses the quantitative demands of a "core problem" to provide meaning and purpose for mathematics and the learnings thus acquired are "broadened, extended and applied." They are not limited to "those problems which arose in the pursuit of some child-selected activity" as was the case in the Hanna study and in this "extension" and "broadening" of learnings beyond the actual "opportunities-found-in-activities" it may be that Miss Williams is using the "guide or criterion," the finding of which was considered "essential" by the Hanna Committee. Beyond the large block of time devoted to core activities, this committee also emphasized the desirability of providing "sufficient periods of practice to assure mastery." This general principle seems to be supported by Miss Williams for in the weekly schedule of her students three 45-minute periods and two 30-minute periods are devoted to "mathematical experiences." Through such planned and creative guidance it is not surprising that the students participating in this problem consistently showed desirable growth in both reasoning and computational ability.

An interesting and significant study on the high school level is reported by Wallace P. Mannheimer.[15] The core problem was "Health" and was designed to integrate functional learnings from English, general science and elementary algebra. Formulas relating to temperature and the multiplication of bacteria were considered while statistical methods were applied to health data. Mr. Mannheimer reports, however, that:

> in our statistical studies we found most of the data formidably complex. To discern trends in such data is a problem to tax an expert statistician, let alone a class in Algebra 1. New concepts can best be learned in simple settings; but the core prevented our doing this. Eventually I was obliged to throw the health figures overboard and substitute very simple data. When I was ready to return to the more complex picture, my colleagues were ready to leave it. . . . The core became a restraint rather than a stimulus to the motivation of algebra.[16]

He states that he made sincere efforts to carry out the "spirit of the core" and re-

[12] *Ibid.*, pp. 118–119.

[13] Williams, Catharine, *The Contribution of an Experience Curriculum to Mathematical Learning in the Sixth Grade*, Unpublished doctoral dissertation, Ohio State University, 1947.

[14] *Ibid.*, p. 15.

[15] Mannheimer, Wallace P., "Mathematics in the Core Curriculum," *Highpoints*, Volume XXVI, No. 8, October 1944, pp. 71–73.

[16] *Ibid.*, p. 72.

ceived the heartiest cooperation from his colleagues in the English and science departments. As a result of this experiment, however, he expresses the opinion that "the 'coring' of mathematics was fruitless" and recommends that "we enrich our teaching of mathematics by relating it, far beyond the boundaries of any core, to the physical, biological and social phenomena about us."[17]

These three experimental situations serve to define a number of important problems concerning the effectiveness with which mathematics can be taught inside the core curriculum. Among these are the following:

1. How is the interest in the core problem affected if the students are required to learn with insight and understanding the mathematics essential to its study?
2. Can the core program provide opportunity for the development of mathematical understandings and skills commensurate with modern living?
3. If core activities are to be "organized without reference to conventional subject lines" what is the effect on mathematical learnings?

One of the attractive features claimed for the core curriculum is the opportunity which it provides for children to participate in the selection of problems in which they have a real interest and about which they are genuinely concerned. The Hanna Committee, for example, refers to the "problems which arose in the pursuit of some child-selected activity" while Miss Williams reports that "active participaton by the children in the selection of their unit of study was an integral part of the school's program of democratic living."[18] Interest is indeed a powerful factor in any educational program and every thoughtful teacher recognizes the value of plans and procedures which claim the interest of his students. This same important factor operates in the learning of mathematics and it might be well to consider what happens when the student's progress toward the solution of the core problem is deflected by the necessity for learning some mathematics along the way. Does his assumed interest in the core problem provide the needed motivation for a genuine understanding of the mathematics essential to its solution or he is tempted to look for answers regardless of how they are secured? When he is faced, for example, with the necessity of multiplying $1\frac{1}{4}$ and $3\frac{3}{8}$ or of finding the circumference of a circle with a given diameter, as was the case in the Hana study, is he willing to submerge the larger drive stimulated by his interest in the core problem and make the kind of detour into the land of quantity and space which is essential if he is to have a real understanding of the processes by which these needed results are secured? Will he have the opportunity to use tangible materials and actually "see" with his hands and with his eyes what the product of two fractions really means before any effort is made to intellectualize the process? Will he be guided in an exploration of the relation between the circumference and diameter of a circle through which he finds an approximate value of pi? Will time be provided to acquaint him with the many significant episodes associated with the evolution of this very important ratio or will his interest in the larger problem call so loudly for the needed answers that he will be encouraged to resort to meaningless manipulation to secure them? How can mathematical meanings be developed, important concepts clarified and general principles established in an atmosphere which places a heavy premium on the importance of answers?

Through such an emphasis the student will tend to believe that mathematics is nothing more than an incidental tool, irritating in its necessity and a heavy burden on the memory. In fact, there is more than a little truth in the statment of Mr. Mannheimer that in any such attempts to "core" mathematics we may be stressing only "its most superficial connections with the rest of the curriculum"[19] and to take

[17] *Ibid.*, p. 73.

[18] Williams, *op. cit.*, p. 29.

[19] Mannheimer, *op. cit.*, p. 73.

the time needed for the building of real mathematical insight is seriously to interfere with the larger drive. A boy who really wants to go to the airport would be quite unhappy if he were first compelled to learn the operating principles of the automobile which took him there. If he does have a real interest in aviation, it is likely to be seriously dulled if, in pursuit of this interest, he is compelled to learn the operating principles of even the most elementary mathematics inherent in the development of this great industry.

The amount and kind of mathematics directly or even indirectly rrelated to a core problem are variables with a wide range of values. The two most important factors which have direct influence on this situation are:

1. The nature of the problem selected for study.
2. The background of the teacher directing the study.

It is possible, for example, to develop an interesting and fruitful study of aviation without including any mathematics and that is precisely what was done by a teacher who himself had no mathematical background. His attitude toward this important field of knowledge does not differ greatly from that of many teachers who are providing very worthwhile experiences through the core program but who are quite unprepared to develop the meaning of even the most elementary mathematical concepts inherent in the study. On the other hand, the study by Miss Williams provides reliable evidence of what can be done by a teacher who recognizes the extent to which mathematical concepts permeate the fabric of modern culture. A less gifted teacher could have guided the students in a very satisfying study of "South America—Past and Present" with little, if any, emphasis on mathematics. Miss Williams, however, greatly enriched this unit through her own appreciation of the value of mathematical concepts and her skill in developing problem situations associated with these concepts. In the preparation of such materials, she used with considerable effectiveness ideas and topics related to South America but it should be emphasized that not all of the mathematics thus studied was *essential* to the unit and not all of it was "based entirely on direct firsthand contacts with reality." Her planning included definite provision for mathematical learnings and the usefulness of the five periods each week dedicated to "mathematical experiences" should not be overlooked. The activities of any core will inevitably reflect the interests and competencies of the teacher guiding the program, and there is no assurance that the core curriculum, as suggested by Professor Fehr,[20] will necessarily include "that mathematics which is needed by all members of a democracy for everyday intelligent evaluation of their lives." The core can be helpful in emphasizing the importance of mathematical concepts and principles and it is quite possible that core activities can provide a readiness for the study of these ideas. The daily schedule, however, should include a special period dedicated to this purpose if meanings, insights, understandings and generalizations are to be made available for wide use. Mathematics teachers should, of course, "contribute wisely and authoritatively to the proper execution of a core curriculum"[21] but there is likely to be a healthy difference of opinion as to just what is meant by its "proper execution." There are, in fact, large elements of chance operating in the selection and conduct of these core units and since an "intelligent evaluation" of responsible citizenship in our modern industrial democracy calls for mathematical literacy, will we be faithful to our obligations if we leave to chance the development of a mathematical program appropriate to general education?

The characteristic of the core which places even more serious limitations on

[20] Fehr, Howard, "Socializing Mathematics Instruction," THE MATHEMATICS TEACHER, Vol. XLI, No. 1, January 1948, p. 7.

[21] *Ibid.*

its usefulness as an instrument for the effective teaching of mathematics is the demand that learning activities be organized "without reference to conventional subject lines." Mathematics is a system of ideas and to make it seem nothing more than a group of discrete manipulative skills, unrelated and unorganized, is to limit its usefulness and to distort its meaning. In a penetrating discussion of "Recent Trends in Learning Theory," Professor McConnell states:[22]

> In the case of arithmetic, attempts to psychologize the subject appear to have damaged it both logically and psychologically. By decomposing it into a multitude of relatively unrelated connections or facts, psychologists have mutilated it mathematically, and, at the same time, by emphasizing or encouraging discreteness and specificity rather than relatedness and generalization, they have distorted it psychologically. They have obscured the systematic character of the subject, and have created a doubtful conception of how children learn it.

How can the potential learnings in a given situation ever become available for wider use unless they are organized around some unifying principle and generalizations established? Mr. McConnell further states that "as learning proceeds, one should construct more inclusive and systematic organizations of ideas and processes governed by the fundamental structure of the number system."[23] Surely no one would support the idea that unorganized experience is educative. What does one really learn from activity alone? The most active core classroom provides no greater guarantee of desirable learning than does any other kind of a classroom. Learning that is worth anything calls for organization of experience and if that experience is organized "without reference to conventional subject lines," it is very difficult to see how mathematical learnings can be expected. According to Professor Buswell[24] "abandonment of organized subject matter cannot be defended" and Professor Katona,[25] after pointing out that the most adequate forms of organization were based upon intrinsic relations, states that "the organization of a series of numbers according to a principle is an example of the kind of grouping which seems most effective for learning and delayed recall."

There are undoubtedly large values achieved through the organization of core activities "without reference to conventional subject lines" and the importance of these outcomes cannot be discounted. To assume, however, that such procedures are effective in the learning of mathematics is to contradict the authority of experience. From the influential pen of John Dewey,[26] whose theories of education have exercised a large influence on the nature of the "core curriculum," comes the statement that:

> . . . it is as true of arithmetic as it is of poetry that in some place and at some time it ought to be a good to be appreciated on its own account—just as an enjoyable experience, in short. If it is not, then when the time and place come for it to be used as a means or instrumentality, it will be in just that much handicapped. Never having been realized or appreciated for itself, one will miss something of its capacity as a resource for other ends. . . .

By what peculiar process of learning can mathematics ever come to be recognized as "a good to be appreciated on its own account" if it is decomposed into a "multitude of relatively unrelated connections or facts"? How can any student ever capitalize on its great "capacity as a resource for other ends" when he has been denied the generalizations which make transfer possible? There is no assurance that mathematical skills used successfully in one situation will be available for use in

[22] McConnell, T. R., "Recent Trends in Learning Theory," Chapter 11, *Arithmetic in General Education*, 16th Yearbook, National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College Columbia University, 1941, p. 275.

[23] *Ibid.*, p. 273.

[24] Buswell, Guy T., "The Function of Subject Matter in Relation to Personality," Chapter II, *Arithmetic in General Education.*

[25] Katona, G., *Organizing and Memorizing*, New York, Columbia University Press, 1940.

[26] Dewey, John, *Democracy in Education*, New York, The Macmillan Company, 1930, p. 281.

another situation. Generalized understandings are needed and the development of mathematical understandings of this kind would be greatly impeded, if not altogether prevented when learnings are organized "without reference to conventional subject lines."

One of the great strengths of the core program is found in the classroom procedures associated with this curricular development. Pupil participation in the defining of problems, the gathering of relevant data, the organization and interpretation of these data as well as formulating and validating hypotheses are educational experiences in a very real sense. But these are procedures which may also be used with great effectiveness by teachers of mathematics and one way in which we can cooperate "in the proper execution of a core curriculum" is to provide for such activities in one's own classroom. Mathematics is in the "core" by virtue of the fact that it is "basic for all students." Let us teach it in such a manner that it is recognized by both students and faculty as "a good to be appreciated on its own account." Generalized understandings are essential if "its capacity as a resource for other ends" is ever to be appreciated and generalized understandings are the outgrowth of experience rather than the basis for it.

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Enrichment of Mathematics Teaching

By WALTER H. CARNAHAN
Mathematics Editor, D. C. Heath and Company, Boston, Mass.

ONE day I was riding along a street with the principal of one of the high schools of a mid-west city. The traffic signal flashed red and I stopped my car at the intersection. The five o'clock pedestrians crossed in front of us. Down from the curb stepped a long, stooped man wearing a crumpled suit, a battered hat, and unpolished shoes. He was not old, but he seemed scarcely an integral part of the life about him. He looked very much like a man with whom I had once attended college. "Look at that man with the tie hanging below his collar," I said. "He looks like a man I once knew." "His name is John White," said my friend. (I substitute this fictitious name for the true one.) "He was once a teacher. Now he is a struggling third-rate lawyer." "John White," I repeated. "Then it is my old school mate."

When the traffic signal changed, we drove on and my friend told me about John White. "Some years ago he was a mathematics teacher at Daneville High School. We needed a teacher, and I made a trip to Daneville to see White teach and to interview him. The lesson I saw him teach was mechanically the most perfect teaching I have ever seen, before or since. Logically it was flawless. He followed every good rule in the methods books. But there was not a spark of life during the whole class period, not a suggestion of anything that had any material existence, not a hint that any human being had ever had any part in the development of the ideas with which he was dealing, not an intimation that anyone ever used the subject for anything at any time. I did not hire him, and in a year or two he quit teaching or was dismissed, I never knew which. Now he is some kind of a lawyer living as you have seen."

Naturally one feels sorry for the John Whites. However, it is those whom they teach who should receive most of our sympathy. They are studying a subject second to none in its natural interest, in its possibilities for material exhibition, in its multiplied intimate connections with every phase of life, and in its dramatic human relationships. It is a subject excelled by none in the possibility of complete satisfaction of the human desire to understand meanings, to go to the very bottom of every concept, to get a final answer to every question of why and wherefore, of "How does this come about?" and "whither does it lead?" The frequent failure of mathematics to interest pupils is not the fault of the subject; it is the fault of the John Whites who teach it; my fault (may I be forgiven!) for having so often and so long taught the subject so mechanically; my fault (and yours?) that teaching has not been adequately enriched.

We teachers hear it said repeatedly that "mathematics trains the mind"; "mathematics develops the ability to think"; "mathematics cultivates the logical faculties." These statements are true, and only good can result from their frequent repetition provided we do not cultivate the delusion that they are the whole truth. This delusion would lead us to develop the concepts of mathematics merely as exercises in logic and to train pupils to prove propositions and solve problems in the same manner. In fact, one sometimes sees teaching done in this very way, forgetting that young learners are creatures of sense and emotion quite as much as of intellect. They like to make things, to look at them, to handle them, and to have them admired. They like to talk as well as to think. They like to know about the people who have worked and contributed as well as to understand their intellectual achievements. They want to know meanings as well as facts. They want to see things

whole and in their natural settings. In mathematics, as well as in history or in domestic art, this is true. The failure of many a teacher to realize his highest possibilities can be traced to his failure to recognize these fundamental needs of learners, his failure to enrich his teaching of mathematics. Human beings are endowed with five senses as well as with minds, and the most effective approach to minds is by way of the senses. Sense and intellect are not mutually exclusive and, properly used, they are not in conflict. The wise teacher makes systematic use of seeing, hearing, touching, making, and using as means of learning and understanding.

Fifty years ago, in order to obtain and retain the interest and cooperation of high school mathematics pupils it was sufficient to say, "Colleges require that you master the subject in order to be admitted." Today that appeal is no longer sufficient because four out of five high school pupils have no prospect of going to college. Once we could secure application to the study of high school mathematics by saying, "It is needed for the study of science or engineering." Today that line has lost its effect. "So what?" they say. "Who wants to be a scientist or an engineer? I am going to get a job in a factory and make some real money." It is true and right that most colleges require mastery of some mathematics for entrance; it is true that it is an indispensable preparation for scientific and technological training; but, as has been frequently said, *education is not only preparation for life; it is life.* Consequently, enrichment of teaching is enerichment of life. And that is the ultimate argument for and justification of enriched teaching. Help the learners to see mathematics as a part of their life experiences; in this way it becomes an integral part of the culture of the people, as it should be.

Even in the days when motivation was a much simpler matter than it is at the present, there was a real loss to learning by failure to enrich the mathematical experience of pupils in every possible way. For enrichment is not to be encouraged wholly nor chiefly as a stimulation of interest, but for the sake of better learning, for the educational benefit that comes from seeing a subject in its correct setting, from seeing it at work, from seeing it whole.

During the past ten or fifteen years, the educational benefits of enriched teaching of mathematics have received ever increasing consideration; ways and means of providing enrichment have been discussed at some length; new materials suitable for providing enrichment have been developed. However, we should be in error if we should believe that the idea of enrichment of mathematics teaching is wholly new.

In fact, the practice of enrichment is probably as old as the teaching of mathematics. Twenty-five hundred years ago, Thales taught the concept of similarity by using a walking staff, a monument, and their shadows. Fourteen-hundred years ago, Aryabhatta wrote his problems in poetry so that sound and rhythm would motivate learning. For example,

Beautiful maiden with beaming eyes, tell me, as thou understandest the right method of inversion, what is the number which multiplied by 3, then increased by $\frac{3}{4}$ of the product, divided by 7, diminished by $\frac{1}{3}$ of the quotient, multiplied by itself, diminished by 52, the square root extracted, addition of 8, and division by 10, gives the number 2.

Four hundred years ago, Billingsly enriched the teaching of geometry by including in his textbook on the subject unique paste-in figures which could be raised to form actual three-dimensional models when the book was opened.

Enrichment of mathematics by means of puzzle and challenge problems is an old and widespread practice. More than six thousand years ago the Egyptians used this one:

There are seven houses; in each are seven cats; each cat kills seven mice; each mouse

would have eaten seven ears of spelt; each ear of spelt will produce seven hekats of grain. What is the total of all these?

The above illustrations are given only to show that the enrichment idea is not new and untried but that its values have long been recognized. Nor was it a practice in which thoughtless persons engaged merely from love of amusement; it was justified by so great a philosopher as Plato who wrote: "There should be calculations specially devised as suitable for boys, which they should learn with amusement and pleasure. . . . There should be games with bowls with gold, bronze, and silver mixed together."

In spite of the sanction of ancient and present philosophy and practice of enrichment of mathematics teaching, certain cautions need to be observed. First, enrichment should be an integral part of all mathematics teaching, not a special feature added on as an afterthought, or substituted for other class procedures, or served up as one serves up a party refreshment. Second, enrichment should not and need not be made to consume much class time, say an average of twenty minutes a week. Third, enrichment is not a substitute for good methods, as the term is commonly used. Enrichment can never successfully substitute for careful, meaningful development, for practice, for testing, for class organization, or for teacher personality. These factors also are time-honored and philosophically sanctioned, and no amount of enrichment can supply a deficiency in any of them. All advocates and practitioners of enrichment agree on this fact. The reader is asked to keep these cautions constantly in mind during consideration of the discussion of certain aspects of the program of enriched teaching of mathematics which follow.

In considering what can be done to enrich mathematics teaching, first concern, of course, should be for mathematics as a science, not for enrichment as a teaching device. Where is mathematics used? Who discovered its principles? Where is it manifested in nature? How can I reveal it as a functioning, inescapable force? These are the important questions rather than "What can I do to stimulate and interest my pupils?" In other words, enrichment is an obligation which one owes to mathematics, to good learning, and to complete comprehension; it should have nothing to do with showmanship or busy work.

It is now in order to give illustrative answers to the questions raised in the preceding paragraph: Where is mathematics manifested, used, and applied, and how can the teacher reveal it to the senses as well as to the minds of learners?

Mathematics is revealed in nature. A drop of rain is a nature-made sphere. You cannot catch a drop of water and show its spherical form to pupils? Well, no. Try a drop of molten lead; it will serve just as well. The lead solidifies as it falls through the air and into a tub of cold water and the sphere is preserved. For a penny you can buy a dozen BB shot and show a natural sphere. Or have Harry bring his bottle of plastic bubble-maker and show the class how a perfect sphere is instantly formed by nature. Have Sarah take an old croquet ball to the basement at home and saw it into two hemispheres, place thumbtacks at the center of the hemisphere and of the section, then, in the presence of the class, wind a string about the tacks, one at a time, and by measuring the strings that cover the surfaces show that one area is twice the other. Also have her show the circle which is a section of the sphere. Get a picture of the crown of spheres formed by a falling drop of milk from *Scripta Mathematica*, New York, and post the picture on your bulletin board. A pearl is a nature-made sphere; an orange is another; as are mother earth, the sun, the moon, the stars and so on.

A snow flake is a geometric figure usually hexagonal. When the snow falls, your pupils would be glad to take cold black paper outside and catch the flakes for study. Your camera fan may be able to get and enlarge for your bulletin board a photograph of a

flake. An unbattered salt crystal is a cube. Have Mary dissolve some salt in warm water, then put the container on a shelf until the water evaporates. The remaining crystals are beautiful geometric figures. An alum crystal is a perfect geometrical figure of a more complicated form. Let Fred find and exhibit one. Rock candy, crystallized sugar on the top of overcooked jelly, and many rock formations are easily obtained geometrical forms from among the hundreds which nature makes.

Leaves on stems frequently arrange themselves according to a simple mathematical formula. The Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13. . . . The nth number equals the $(n-1)$st number+the $(n-2)$d number, the series beginning with 0, 1. Form the ratio of any number to the following number, and (at least up to $\frac{8}{8}$) it is easy to find leaves which spiral up the stem by making the fractional part of a full turn indicated by the Fibonacci ratio. An appropriate place to introduce the Fibonacci series is in connection with the Golden Section in plane geometry, because the Fibonacci ratio approaches as a limit the ratio of the Golden Section as the size of numerator and denominator increase without limit. The spiral formed by sunflower seeds is also intimately related to the Fibonacci ratio. Have Bill bring a sunflower to be hung up as a mathematics exhibit.

A snail shell or a sea shell shows a logarithmic spiral. Have John bring a shell and show how to obtain its form by graphing $y=e^x$. A good time to do this is when the intermediate algebra class is studying logarithms. Get Baravalle's figures showing the logarithmic spiral made by arrangement of geometric forms.

These examples are but a small introduction to the abundant mathematics of natural forms.

Mathematics is in all modern scientific discovery. On September 23, 1846, John Galle in Berlin received a letter from Urbain Leverrier in Paris. Both men were astonomers. Leverrier had firm mastery of mathematics but no telescope. Galle had access to a good telescope but a less profound knowledge of mathematics. Leverrier's letter said in substance: "Turn your telescope to the exact spot in the sky to which I direct you and there you will see a planet never before seen by man." That night at the exact spot indicated, Galle found the planet Neptune. No book that I have read tells whether Leverrier ever saw Neptune, and yet he is always credited with its discovery. This was an achievement of mathematics. He used facts available to every interested person, scientific laws which could be read in a hundred books, and mathematics.

Forty years ago, Albert Einstein announced the equivalence of matter and energy by means of the formula $E=MC^2$. (Read his surprisingly simple development of this formula.) This formula says that a kilogram of U235 contains 900,000,000,000,000,000,000,000 *ergs* of energy. Einstein operated no cyclotron, bombarded no atoms, dropped no atomic bombs. Mathematics alone (applied to natural laws) enabled him to establish the relation which ultimately led to the atomic age.

You can add without limit your own illustrations of mathematics in scientific discovery.

Mathematics gives the answers to many profound questions concerning GREATEST, LEAST, *and* CONSTANT. A ball is to roll from point A to point B in the least possible time. What is the form of the path which it must take? Mathematics gives the answer. As a forceful material reminder to pupils of the power of mathematics to answer questions, have Ted cut a grooved *cycloid* in a board and display the *brachistochrone*, the curve of quickest descent.

A path is to be made along which a ball rolls in the same time whether it traverses the whole path or one tenth of it. Is there such a path, and what form should it have? Mathematics has the answer. Put before your pupils in permanent material form the *tautochrone*.

A bearing is required which will never lose its shape, no matter how much the contact surfaces wear away. What should be its form? Have Arthur turn out on his lathe a model of Schiele's pivot and show the answer which mathematics gives to such practical questions.

Mathematics enters into many of our amusements. A frog is in a well which is twenty feet deep. He jumps upward three feet each day and slips back two feet every night. How long will it take him to get out? This problem has interested men for hundreds of years.

A man died and left seventeen camels to his three sons. The eldest was to inherit one half of them, the second was to have one third, and the youngest was to have one ninth. How can the division be made? How many persons have spent how many thousands of pleasant hours with this trick problem during the hundreds of years of its history?

How many men have found surcease from care, inflationary worriers, and fear that Junior would put a dent in a fender of his new car by spending an evening with the old problem of the five men, the monkey, and the cocoanuts? How many persons have composed themselves for sleep by trying to solve a unicursal problem, or by marking and cutting a Mobius band, or by trying to find the age of Ann? All of these puzzles are "Just for fun," but they are also mathematics. Do you reserve a corner of your bulletin board for the posting of such problems?

Mathematics is in art and decoration.

Mathematics is in advertising, in propaganda, in the Kinsey Report.

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The teacher who wishes to enrich the learning experiences of his pupils needs to spend time in enriching his own knowledge of the subject. Some of the time which she might prefer to devote to bridge, or to parties, or to entertainment, or to reading the book of the month, or even to taking college courses in mathematics might be profitably spent in reading *Mathematics and The Imagination* by Kasner and Newman, or *Riddles in Mathematics* by Northrop, or *The Education of T. C. Mitts* by Lieber, or *Growth and Form* by D'Arcy Thompson, or any one of a score of other such books. The *Eighteenth Yearbook* of National Council lists many books which the teacher can use to enrich her own experience with mathematics and that of her pupils.

The teacher who puts into effect a program of enrichment of mathematics teaching may have the experience of finding that pupils become fascinated by the enrichment experiences and fail to transfer this interest to the mastery of the principles and techniques of arithmetic, algebra, or geometry. It is by no means the easiest phase of enriched Mathematics teaching to make it function as a true educational experience, but this must be done or the whole program will fall into disfavor and we shall lose many of its very real benefits.

To Members of the National Council

You can help the Council greatly by seeing to it that the Guidance Pamphlet in Mathematics is given more publicity. It would be a fine thing if at each meeting of mathematics teachers, someone could be delegated to have some of these pamphlets for sale. THE MATHEMATICS TEACHER will be glad to send on assignment whatever copies are needed.

A Course in College Mathematics for a Program of General Education

By C. V. Newson
University of the State of New York, Albany, N. Y.

The factors involved in the educational process are so complex that it is difficult to find agreement among thinking men upon fundamental matters of educational policy. Thus we have a multitude of plans and movements in the field of education; this seems to be especially true during the past couple of decades, perhaps because of the growing complexity of life itself. However, a close inspection of the situation in American education seems to reveal one trend of recent years that is undeniable; that is, there is remarkable agreement on the fundamental proposition that specialization should not start too early, that a broad background in the various fields of knowledge is prerequisite to genuine success in any one narrow field of endeavor.

Under such a circumstance, it is inevitable that the content and purpose of the traditional courses of study in our colleges and universities should be reviewed to see to what extent they fulfill the demands of a broad program of general education. The problem is difficult, for there is no agreement upon the meaning of the term "general education." Moreover, the English professor or the economics professor, for example, will express opinions about the objectives of the mathematics course, just as the mathematics professor believes that he has a right to be heard upon the purposes of a course in history. The situation is further complicated by the fact that the educational specialist believes that he is in a position to take an over-all view of the problems involved, so he expresses his views on English, mathematics, and history, as well as on some new disciplines not found in the curriculum of twenty years ago. Some of the resulting arguments have generated considerable steam and bitterness, but it is doubtful that we are observing any more than the growing pains of a body of educational philosophy that ultimately will enable us to realize many outstanding accomplishments in American education.

The story of the development of a course in mathematics that can become a part of a program in general education has been told by many writers. K. E. Brown made a notable contribution to the literature of the field when he published his book, *General Mathematics in American Colleges*,[1] in 1943. This book contains excellent bibliographies, and outlines the evolution of the general mathematics movement and the various objectives that have been formulated. There is little reason to repeat the story that Brown tells. Rather, it seems desirable, by way of introduction, to report some personal experiences that might be regarded as typical of those of any college teacher of mathematics who wants to be of maximum service to his students. Of course, other teachers might try to solve the perplexing questions that were a part of this personal experience in an entirely different manner; that fact keeps alive the discussion of general education, and no unique solution of the many problems involved should be anticipated.

In 1931, it seemed desirable to review the entire program of mathematics at the University of New Mexico. Data were compiled and examined, and opinions were sought from several hundred graduates of the institution. Among the interesting conclusions revealed by the survey was the fact that, outside of the college of engineering, over 80% of those students who enrolled for any college mathematics took no more than one year of work in the

[1] Bureau of Publications, Teachers College, Columbia University.

field. The course that had been available to them was the traditional offering of college algebra, trigonometry, and analytical geometry. Were the 80% satisfied with such a course? No! The replies from them were almost unanimous in asserting that the freshman course in mathematics had taught them nothing about the real meaning or even the utility of mathematics. Most of the alumni stated that, in retrospect, they regarded the course as "a waste of time," and they had a vague recollection of juggling involved algebraic expressions and getting numbers out of tables for the solution of triangles. One of the most bitter letters came from an educator who was then leading a fight in opposition to mathematics; perhaps other opponents of mathematics have a similar background of experience in the field. It may be stated, parenthetically, that the same students might be happier now with the same sequence of courses, as there has been considerable change in textbooks during the past twenty years. The apparent urgency of doing something for the 80% was enhanced by the fact that preliminary discussions were inaugurated in the University pertaining to the creation of a "general college" that would give a two year basic program of studies.

The experiments and discussions that took place in the mathematics department for the next few years were both interesting and informative. It seemed originally that any single course in general mathematics should have a unifying idea, and, quite naturally, the notion of function was selected as the principal theme. Some simple statistical notions were taught, but the greatest emphasis was given to the studying of functional relationships, especially empirical formulas and their curves. The course was notably unsuccessful; the concept of function possesses inherent difficulties that are bothersome to the beginner, and the students generally did not have sufficient background, even in arithmetic, to handle such material.

In an attempt to remedy deficiencies in arithmetic, postulates for the number system were set up and examined, and the axiomatic method was analyzed. It is always difficult in the development of such a course to distinguish between the enthusiasm of the students and that of the teacher; it is probable in this particular instance that the teacher played the dominant role for a few years as the course went more and more in the direction of a study of the foundation of mathematics. A course now being taught in a leading American university resembles so closely the course taught at the University of New Mexico in 1935 that even the order of topics is the same; it is unfortunate that there is so little exchange of information and that there must be so much duplication of failure. When the dean presented preponderant evidence that the course as constituted was not popular, a new evaluation of the program seemed desirable.

No one would deny that the instructor of a course is in a much better position than the students to understand what kind of mathematics is fundamental from a long-range point of view and how it should be taught, but an open-minded study of our difficulties revealed to us that we were failing because we were ignoring almost entirely the immediate interests of our students as well as their mathematical immaturity. It is my firm belief that any sucessful course in general mathematics must start with a study of arithmetic; the same point of view has been expressed by C. B. Allendoerfer.[2] Students with four years of high school mathematics and with excellent grades may know many of the mechanical devices employed in arithmetic, but, in general, they know nothing about the number system and the meaning and inter-relations of the arithmetical operations. This comment should not be regarded as a criticism of the instruction in the elementary schools and

[2] *Mathematics for Liberal Arts Students*, American Mathematical Monthly, Vol. 54, No. 10, Dec. 1947.

high schools; there comes a time in the life of a student when he appreciates the need for synthesis and more thorough understanding; it is doubtful that such a period is reached much before college. Also there is a limit to the typical student's ability to enjoy too much pure abstraction; theory without application is for the very few. Students in a general course are not content with hearing statements about the utility of various concepts; they want to be participants; they want to apply their knowledge to actual problems. Unfortunately, the meaningful problems familiar to most teachers of mathematics are in the physical sciences, where most students in a general course have decidedly limited background. A fertile source of useful and interesting problems is the field of finance; the dollar sign is the symbol for something very real for most students. In spite of the difficulty inherent in even comparatively simple problems in probability, permutations and combinations, the teacher can hardly afford to overlook the subjects; students thoroughly enjoy the topics if significant problems are considered. The concept of variation is as important in the social sciences as in the physical sciences, and problems involving the notion lead to some interesting mathematical experiences. Virtually all fields of knowledge yield meaningful problems if a diligent search is made for them. Among the physical sciences, problems from astronomy seem to have the greatest appeal.

In short, the members of the staff began to recognize that some measure of success was attending their efforts after the course was completely reworked so that its level of presentation was determined by the maturity of the students and when some balance was achieved between theory and application. The University of New Mexico, in common with many institutions, had a group requirement that specified that students must take a definite amount of science or mathematics; a very large proportion of the students in the institution began to select mathematics to satisfy, at least in part, this requirement. Within a few years, moreover, at least half of the students in the course had decided to study it under no compulsion whatsoever, and the large number of upper classmen taking the course was noticeable; most of these latter students were majoring in the social sciences and in the humanities. Every section included a large number of auditors, even faculty members. No other course that I have taught has given me so much personal satisfaction.

During the summer of 1939, the writer of this article had the good fortune of being appointed chairman of the college mathematics group that met at the University of Chicago under the auspices of the Cooperative Committee for General Education. The group was composed of representatives from several mathematics departments, and down-to-earth discussions took place in regard to teaching experiences, and an analysis was made of successes and failures. It is to be hoped that the members of the group learned as much as the chairman; certainly the meetings were highly profitable. It is my personal belief that the National Council of Teachers of Mathematics, as well as the Mathematical Association of America, could do much toward the stimulation of good teaching if similar two or three weeks conferences were held each summer; the discussion groups should be kept small, and each group would limit its attention to one major aspect of the teaching program.

Many institutions have achieved similar success in the creation of a one-term or a one-year course in mathematics for the purposes of general education. These courses may take different forms according to the general educational philosophy sponsored by the institution or because of some special need of the students. But in spite of the accomplishments, many questions remain to be analyzed and many problems remain to be solved.

In the first place, the content of a course

in general mathematics needs to be studied on an elaborate scale. As indicated previously, it is my firm belief that a successful course must start with a consideration of arithmetic; beyond that, it appears that there are many possible topics and concepts that might be considered. The important concept of function with its many ramifications should be a part of the course, but the failures in presenting the idea vastly outnumber the successes; the pedagogical problem involved is sufficiently serious that many scholars interested in the teaching of mathematics should be working on it. The important topic of measurement and approximate numbers should be taught in some detail, but the writer of this article has looked in vain for a treatment of the subject that does much more than mystify the beginning college student. Fortunately, the arithmetic of measurement is being considered by mathematicians on every level, and there is reason to believe that a sound approach to the subject will be developed in the near future. Any course that proposes to give the students a glimpse of the powerful methods employed by mature mathematicians should certainly give some attention to the calculus. Again, however, it is my considered belief that few if any of the brief introductions to the calculus included in most survey courses provide the students with any appreciation of the importance of the subject. The teacher may succeed in teaching a few of the mechanical aspects of the calculus, but that is the very thing that should be avoided in the type of course under consideration; the mechanics of manipulation must proceed hand-in-hand with understanding, one without the other is of no value. The question pertaining to how and when the calculus can be introduced as a part of a general course represents a challenge to the good mathematician who is also a good teacher. In this age when every person must deal with phenomena that involve some form of periodic motion (that is, radio, light, music, and so on) it is a mistake to ignore the trigonometric functions, but the trigonometry that is envisaged could completely ignore the solution of triangles; triangle solving has little significance for most people. Also, a program in general education can no longer ignore the universal acceptance of statistics as fundamental in almost every phase of human endeavor; departments of mathematics must accept responsibility for teaching this subject. In a general course in mathematics, the instructor must not only teach the significance and proper conduct of statistical analysis but must warn of the dangers attendant upon haphazard use of statistical methods; the conclusions that are being foisted upon an innocent public by some so-called statisticians are so unjustified that every elementary presentation of statistics should spend some time on fallacious statistical reasoning. Allendoerfer would teach an entire term of statistics,[3] but many would doubt that such a large proportion of the general course should be devoted to one topic.

A fundamental problem in the construction of a one-year course in general mathematics pertains to the fact that some students will become interested in the field of mathematics, and will want to take advanced work in the subject. Even though the enrolment in the course is restricted to students who signify that they have no intention of taking more than one year of mathematics, with the possible exception of statistics or the mathematics of finance, it has been the writer's experience that approximately six per cent of the students later announce that they proposed to major in mathematics. This introduces a serious problem inasmuch as a good course in general mathematics will not provide an adequate background for the calculus or the traditional sequence of advanced courses. In general, students completing the general course will be poorly equipped to handle problems requiring extensive algebraic manipulation.

[3] *Loc. cit.*

Of course, if a sufficient number of students indicate their intention of continuing their mathematical studies, a special one-term course can be created to bridge the gap between the general course and the traditional sequence; in case this is not possible, a somewhat difficult situation may arise, but frequently a good course in college algebra will serve the purpose fairly well.

The obvious solution to the difficulty just discussed calls for a revision of the traditional curriculum in college mathematics so that the general course under consideration would merely become a first course for all students. Interestingly enough, much of the material covered in the better general courses is not duplicated anywhere in the traditional sequence so that the student completing the general course will actually have a type of mathematical knowledge and an understanding of mathematical concepts that the typical major in the field does not possess. This was called to my attention in a striking fashion two years ago when a group of senior students preparing to be high school teachers of mathematics called on me to inquire why they should not be given the opportunity to study the type of material that was taught in the general course. Professor E. P. Northrop of the University of Chicago has expressed the opinion that a course in mathematics that is fundamental in the general knowledge of non-mathematicians should be just as fundamental in the general knowledge of the mathematician. Some have expressed the thought that this point of view is intolerable because students taking the general course have mathematical ability less than that to be found among students majoring in mathematics. It is doubtful that this is true; at least, there is no evidence for such an attitude. In fact, a general course can and should be as difficult as the other courses in mathematics; even fundamental concepts of arithmetic can be presented from a mature viewpoint. It is noteworthy that in those institutions that offer a general course the impact of the new approach has generally left its mark on the traditional courses.

The future of the course in mathematics for purposes of a general education is secure. This is due in part to the wide support being given the philosophy of a general education. O. C. Carmichael, President of the Carnegie Foundation for the Advancement of Teaching, has written.:[4]

> Fragmentation of knowledge through emphasis on departmental subject matter has been the least defensible of our educational practices. The required curriculum of the arts college, made up of courses in various subjects designed as the first step in a four-year series looking to specialization, has little chance of contributing to liberal education, and yet that has been the pattern of the average American college. . . . The general-education movement, if directed toward restoring broad, liberal learning and real intellectual interests to the great body of undergraduates, may succeed in correcting some of the serious weaknesses of higher education. There are some indications that this is the direction which it is taking. A leader in the movement who has spent several months during the current year visiting college classrooms reports a surprising interest in strengthening and broadening the content of the undergraduate program.

There are other reasons to believe that a course in general mathematics has become a permanent part of our educational program. For instance, there is a growing realization that mathematics has a unique place to fill in the field of education that cannot be filled by the study of a science. The University of Michigan has inaugurated a new graduation requirement in philosophy or mathematics; reports indicate that other institutions are breaking away from the absurd graduation requirement of a specified number of hours of science *or* mathematics. Also, the increased emphasis on mathematics in the social sciences is causing more and more students to desire a basic mathematical understanding. A very important factor in the growth of interest in all forms of general and terminal courses is the rapid development of junior or community

[4] *Some Educational Frontiers*, School and Society, Vol. 68, No. 1761, Sept. 25, 1948.

colleges. Of course, the general course in college must still run a gauntlet of unjustified criticism and misunderstanding, but every educational reform meets objection when first introduced.

Previous to the last world war courses in general mathematics were becoming quite common in the institutions of this country. However, the prescribed military programs forced a quick return to the manipulative mathematics of a traditional type. Although teachers of some disciplines learned important lessons from wartime teaching, the mathematics teacher learned nothing. Certainly the negative effect of the war on the development of general courses was noticeable. Now, however, interest is again on the upswing. Publishing houses have been impressed with the number of manuscripts being developed in the field. With so many competent mathematics teachers working on the subject, it is to be hoped that there will be an extensive exchange of ideas so that the courses being developed will be of maximum service in a modern program of education.

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Let's Come to Grips with the Guidance Problem in Mathematics

By RALEIGH SCHORLING
School of Education, University of Michigan, Ann Arbor, Mich.

THE unifying goal of a good guidance program is the conservation of human resources. The crux of the problem is first, to identify the basic pattern of a pupil, and second, to put into his environment the particular nurture that his growth needs require. Francis W. Parker expressed the idea neatly a half century ago in the famous quotation, "The fundamental reason why children do not act right is because they do not have the conditions for right action." A successful grower of blueberries is well aware of this basic principle. He has to know a blueberry bush when he sees it, and surely he will have to know a lot about its nurture. For example, if his soil is not sour, he goes into action fast, for he knows that if he does not, his blueberries will soon look very sick. In any event, he does not expect to pick roses off the blueberry bush!

School is not a good place for a student if the setting deprives him of experiences that are essential to his growth pattern. Pupils at different stages of maturation will tend to select, and certainly will thrive on different experiences. The evolving literature on child growth and development emphasizes the pupil's tendency to select nurture in terms of his growth needs. Obviously, then, it is the job of the teacher to assist the pupil in achieving socially desirable goals according to his basic pattern, without force or deprivation.

The security of a pupil is undermined if he is subjected to (a) rejection by his classmates, (b) excessive and unfair criticisms by the teachers, (c) failure on tasks beyond his ability, and (d) faulty classification in school. In the lives of many pupils the gap between the needs of the normal growth pattern and the expectation of family and school is the devastating factor that causes frustration, despair, and rebellion. Students who have a deep sense of insecurity in mathematics are probably not enrolled in the right course.

If the preceding list of idealistic principles for guidance is acceptable, then we can use it as a yardstick and see what needs to be done in order to increase the chances that each student may get the mathematical nurture that he needs. As many experienced teachers of mathematics realize, our main weaknesses may be listed as follows:

1. *Too many pupils take a rigorous sequential course in algebra before they are mathematically mature enough.* They fail in algebra when an additional year of a good course in general mathematics would give them a chance to succeed.

2. *Too many pupils in high school take no mathematics, and later discover that they lack specific mathematical training for what they want to do and can do.* When a student does not have the ability to succeed in algebra it is an oversimplification of the problem when he is advised to elect no mathematics beyond the eighth grade. Either the Check List[1] of 29 items for functional competence suggested by the Commission on Post-War Plans is not valid or such students should take courses in mathematics designed to give them an opportunity to acquire these competencies.

3. *Many pupils fail in formal courses in algebra and geometry, who should never be in such courses.* The national picture of the devastating effect of unnecessary grief

[1] "Guidance Pamphlet in Mathematics for High School Students." Final Report of the Commission on Post War Plans of the National Council of Teachers of Mathematics, November, 1947. THE MATHEMATICS TEACHER, 525 W. 120th St., N. Y. 27, N. Y. Pp. 1–25.

on mental health of pupils and parents is genuinely tragic. Let us for a moment consider the question: Who should study algebra? As a simple guide for high school pupils and teachers, we suggest that the pupil who can do a good job in algebra without too much grief to himself, to his family, and to his teacher, who enjoys algebra, and who is likely to make some future use of it, is the pupil who should enroll in an algebra class. Along with this, the pupil might well be told in his guidance program (a) that few adults, if any, who are competent in algebra have ever regretted time spent in learning the subject, (b) that algebra is an easy subject if a student comes to it with adequate training, and if it is well taught and the structure is built carefully step by step, and (c) that there are few subjects which pupils like better than algebra provided it is well taught.

4. *Too many administrators, supervisors, home room teachers, and other guidance personnel fail to realize the large number of persons needed to do the scientific, professional, or highly technical tasks of modern life.* It is generally admitted that teachers of mathematics in high school, junior college, and university, statisticians, engineers, surveyors, actuaries, physicians, surgeons, physicists, chemists, and scientific researchers need good training in mathematics. However, the very large number of persons involved is not generally recognized. Perhaps it is sufficient to point out that there are over 200 scientific organizations that are affiliated with the American Association for the Advancement of Science. Then too, one should realize that some of these groups are very large. For example, there are more than 175,000 physicians, 67,000 dentists, and 100,000 pharmacists. Research in industry is growing by leaps and bounds. For example, in 1920 there were only 297 research laboratories in industry, but in the next two decades this number jumped to 2264. In 1946 there were about 50,000 scientists in industrial laboratories, plus 37,000 technical workers, making a total of persons who need good mathematical training of about 90,000 for such work alone. Highly significant is the fact that in the school year 1947–48 we had more than 600,000 college students enrolled in one or more science courses. A reading of Volume IV of Science and Public Policy, a Report to the President by the President's Scientific Research Board, suggests the conclusion that if from 20–30% of the high school population were carefully schooled in algebra, that this number would, in all probability, take care of this country's future need for technical and scientific workers. The typical administrator does not realize that the pool of future scientists is a goodly fraction of the high school population whose rigorous training should not be neglected.

5. *The offering in mathematics in the small high school—about two thirds of all our high schools—is much too limited.* Many small high schools do not offer general mathematics, and probably an even greater number do not offer algebra. The offering could so easily be enriched by (a) a cycling of courses, (b) teaching two courses in mathematics at two levels simultaneously or (3) by correspondence courses now serviced by more than 80 universities.

6. *Many good students elect the wrong courses even when a choice is provided by a variety of courses on the double track.* Often this is due to tradition which causes guidance personnel to pressure students to elect traditional sequential courses to the neglect of the mathematics that they need in industry, on the farm, on the first job in businsess, or in managing a home. The false assumption that the only worthwhile mathematics is to be found in the traditional sequential courses and its corollary that general mathematics is designed for the excessively dull, are still operating to undermine mathematics programs throughout the land.

7. *We have not been sagacious in the use of human resources as regards early identification of youngsters with a flair for*

mathematics, and their proper culture. Nor can we be confident that many teachers of the early grades, with little or no training in science and mathematics, can spot a youth with talent or that they would be able to design appropriate experiences to fan the spark of genius. It appears that the ratio of able youth out of college to those in college is about 1 to 1. To be sure, this loss of precious talent is often due to the fact that many youths in normal times can not afford higher education. However, too often the tragedy is due to faulty guidance.

8. *High school teachers in general can not provide their pupils with reliable information about the science and mathematics needed on jobs and in professional careers.* There are two main reasons: (a) our teachers lack knowledge of the applications of science and mathematics made in engineering, agriculture, mining, medicine, and the like, and (b) they do not have the necessary guidance materials.

A New Tool for Guidance in Mathematics

Fortunately, we now have an instrument that should prove enormously useful in providing adequate guidance in mathematics at the high school level. The first guidance pamphlet geared to a high school subject was published in the November, 1947 issue of The Mathematics Teacher The purpose is to describe for high school pupils some of the occupations in which mathematics is important. The pupil learns about the amount and kind of mathematics that is used and the school level at which it should be taken. Reprints should prove very useful to guidance personnel, in classifying pupils. Here teachers of mathematics can find information that they can give to pupils and parents in a long time program of guidance. The hope, of course, is that the choice of the proper mathematics courses in doubtful situations will be made by pupils and parents in terms of opportunity, interest, and ability. Too often pupils and parents are crushed by official decisions which, to them, seem to slam the door to opportunity. As a result, there is great pressure on teachers to permit pupils to take courses for which they do not have adequate preparation.

Too often helpful documents are created by professional organizations, and become "cemeteries of good ideas" for lack of follow-up steps. To avoid this hazard, the National Council of Teachers of Mathematics, and especially Dr. W. D. Reeve, through The Mathematics Teacher, is putting forth strenuous efforts to insure that the Guidance Pamphlet is used widely and in effective ways. The number of reprints distributed to date—more than 25,000 copies in a little less than a year, is truly gratifying.

It is the purpose of the remainder of this article to list some of the specific ways in which teachers of mathematics may make good use of the Guidance Pamphlet.

Let us begin with the most important factor—the classroom teacher of mathematics. The specialist in guidance has a place on a high school staff to instruct his fellow teachers in guidance techniques and to advise on difficult problem cases. However, the role of the classroom teacher in the total guidance program of a school is obviously a most important one. What are some of the specific steps that a mathematics teacher can take in the effort to make a wise use of the Guidance Pamphlet in Mathematics for High School Pupils?

1. He can secure a copy of the Guidance Pamphlet and thoroughly familiarize himself with its contents. As has been suggested, it contains good answers to many of the questions that pupils and parents ask.

2. He can see to it that the principal, his fellow teachers, classroom advisers, and other guidance personnel, know the Pamphlet. Three dollars will buy 30 copies, and it is difficult to think of a better investment for a mathematics teacher. Surely his work will be easier if the right pupils are in the right classes. The most effective way to get the pupils in the ap-

propriate classes is by a continuous guidance program rather than by devastating and hasty decisions stemming from some magic formula of classification. To that end, the classroom teacher would do well not only to see to it that each one of the persons in the building with responsibility for guidance has a copy of the Guidance Pamphlet on his desk, but also to check from time to time to make certain that it is used properly and not forgotten or misplaced. It has been gratifying to note the hearty endorsement of the Guidance Pamphlet by outstanding specialists in guidance, as for example, Professor Ruth Strang of Teachers College, Columbia University.

3. He can, working through his principal, encourage boards of education to buy a set of 30–50 copies for each mathematics classroom as a part of the regular classroom equipment. From time to time a few minutes of a class period can be taken to encourage pupils to read such parts as may be of interest to them. Many schools spend a considerable sum of money each year for materials relating to guidance. The suggestion here is that schools be encouraged to buy the Guidance Pamphlet in the same way that they do other guidance materials.

4. He can, if an adequate number of Guidance Pamphlets is available in the classroom, once each semester, or at least once a year, loan a copy to each pupil and his parents for about a week and encourage them to plan the pupil's future mathematical experiences in terms of the information provided.

5. He can, when he participates in planning a local or regional program for mathematics teachers, make a special effort to see that copies ofthe Guidance Pamphlet are available for examination. One needs to remember that the membership of our profession shifts very fast, and therefore one can not assume that a considerable fraction in an audience of mathematics teachers will have examined the document. Mere information as to where the Guidance Pamphlet can be secured and its table of contents will, in general, not be adequate to stimulate curiosity. Whenever possible, a brief time at least should be devoted to an exchange of experiences in its utilization.

The National Council of Teachers of Mathematics can take steps not only to insure a wide use of the Guidance Pamphlet, but also to improve it.

1. The National Council can make certain that an adequate number of copies are available at each regional, state, and national meeting of mathematics teachers. In advance of the meeting, the chairman should estimate the number of persons likely to attend, and procure enough copies for those who may wish them. An effective procedure is for the chairman to place copies on the speakers desk and suggest that any member of the audience who wishes a copy may pick it up and leave a dime. In a recent meeting of 110 teachers of mathematics, not more than 5 minutes was devoted to the Guidance Pamphlet, but 102 copies disappeared, and $10.20 was left by the audience to pay THE MATHEMATICS TEACHER for the copies taken.

2. The National Council can keep in touch with the "grass roots" as regards the response of teachers, pupils, and parents. What parts are helpful? What is lacking? What sections are not useful? Perhaps a small committee should be appointed to collect and interpret the evidence on a nation-wide basis.

3. The National Council can appoint a standing committee with changing personnel, and instruct it to revise the pamphlet periodically, say every 5 years. Presumably no one thinks that we have the ideal tool for guidance in mathematics, but we do have a beginning. It should prove a very easy task in the future to publish revised and improved editions. Perhaps we can some day publish a larger edition with much new material, enlivened by attractive illustrations, and a vocabulary still closer to the high school pupil, at even less cost.

A Forward Look at Evaluation

By Maurice L. Hartung
The University of Chicago, Chicago, Illinois

Introduction

When a group of teachers sits down to plan improvements in the curriculum, sooner or later someone will remark: "I think evaluation is very important, and we should say something about that, too." To this there is quite general agreement, but unless a group is specifically set up for this purpose, they are likely to become so engrossed in problems of content and method that evaluation gets postponed and superficially treated. Yet emphasis upon adequate evaluation is one of the surest and most scientific ways of improving instruction. The following paragraphs will consider two main aspects of the total evaluation problem. The first part of the paper will suggest three ways in which comprehensive achievement tests can be improved. The second part of the paper will consider some of the problems associated with the use of evaluation data for guidance purposes. The views expressed are offered without documentation and may be regarded as purely personal. Anyone who has given attention to the literature on mathematical education will recognize, however, that similar suggestions by others have been made before and may be found in various sources.

The discussion in this paper will be based upon the following assumptions and definitions. The purpose of education is to produce changes in the behavior of pupils—changes in the ways they think, feel, and act. Statements which describe the kinds of behavior deemed desirable are the objectives of education. The process of obtaining and interpreting data about the development of pupils is called evaluation.

The task of formulating objectives is not, strictly speaking, a part of evaluation. It is, however, of crucial importance in planning the curriculum, since the objectives determine the kinds of learning experiences provided for the pupils. When objectives are being formulated, the value systems of those who do the formulating play a dominant role. The statements of objectives reflect what teachers and others who make them believe to be important or valuable—to be worth doing or learning. Moreover, if maximum success is to result from school experiences, the objectives must be consistent with what is now known about the growth and development of pupils in general, and in particular, how they learn. In other words, the objectives must take into account the principles of anthropology, psychology, and related fields. Finally, to be realistic, the objectives for a given course and group of pupils must take into account the present status of those pupils. That is to say, they must be chosen in the light of data on what the pupils know and can do at the outset. Thus evaluation is needed in order to formulate an appropriate set of objectives and it is essential in order to determine how well the objectives—the desired *changes* in behavior—are being achieved.

At the present time evaluation in mathematics is still limited in scope. The following procedure is characteristic. The "homework" papers or workbooks of pupils are checked and some sort of record kept. Teachers give quizzes and unit tests periodically. These papers are scored and the grades are recorded. At the end of the semester or year, a "comprehensive" achievement test is usually given. Scores on all tests, often weighted according to some arbitrarily determined scheme, are averaged and final letter or numerical grades are assigned. The "passing" mark is also determined arbitrarily. If the tests are local or teacher made, the difficulty of the items is commonly adjusted in ad-

vance so that a sufficient number of pupils pass to minimize the tensions resulting from an "excessive" failure rate.

No space will be taken here for critical comment on the procedures outlined above. It should be noted, however, that some teachers, and they are by definition the better ones, improve upon this routine process by devoting time and effort to careful analysis of the strengths and weakness of each pupil. They use tests diagnostically and they set up learning experiences specifically designed to overcome weaknesses discovered. Moreover, the tendency to encourage pupils to participate in the evaluation is increasing. In these and in other ways the teaching profession is slowly moving toward a better type of evaluation program. The important issue here may be stated as a question: What needs to be done now by teachers of mathematics to make their evaluation program more effective?

Improvements in Achievement Tests

Three types of improvement in tests are within the reach of mathematics teachers in the near future. First, the comprehensive achievement tests which are given from time to time should be built with separate parts which correspond to the important objectives. Thus for many years the popular arithmetic tests have had at least two parts—computation and problem solving. On the high school level the available tests have also commonly had several parts, but too often these have been determined in terms of test techniques used. Thus one section may be composed of completion type items, another of multiple-choice items, and so on. In these circumstances the analysis of achievement in relation to the several objectives has been difficult. The part scores have little meaning as such and are generally not highly reliable. For some time past these weaknesses have been apparent to critical observers. There have been efforts to build tests such that each of the part scores provides evidence of achievement of some important objective. In the future this trend should be strongly encouraged, until eventually tests should have at least four or five such parts. Efforts should be directed toward making each part as independent of others as possible. In statistical terms, the correlation coefficients based on distributions of part scores should be made as low as possible. At the same time, the reliability of the part scores should be increased. Assuming that valid items are available, this can usually be accomplished by lengthening the part. This requires more testing time, but it is not difficult to defend this expenditure of time and effort by reference to investigations showing the improvements in curriculum, methods, and achievement which will normally follow the practices outlined above.

Second, analysis of available published tests reveals that in the case of some important objectives of mathematical instruction achievement is not at present being adequately evaluated by them. The informal geometry now commonly taught in grades seven through nine is an example. Authoritative national curriculum reports have recommended a fairly extensive geometry program for these grades. Good learning materials in the form of textbooks, workbooks, and visual aids are now available. Under ideal conditions the student entering plane geometry courses should be in possession of most of the important geometric concepts, theorems, and skills he will encounter in the course. The course in demonstrative geometry should be largely devoted to learning principles of logic and applying them to these ideas until the student sees how the geometric content can be fitted into a logical system by postulational methods. To make this program effective, teachers should have evaluation instruments which enable them to keep track of the growth of geometric knowledge of the students through the junior high school grades culminating in a precise summary of the status of each student on entering plane

geometry. No evaluation instrument adequate for this purpose is now generally available.

As a second illustration of an objective now inadequately evaluated, the one which deals with understanding of the role of mathematics in the development of civilization and in modern life may be cited. If this objective is important, an adequate evaluation program would secure reliable evidence of growth in this understanding from year to year. At present such growth can only be roughly estimated on the basis of relatively subjective observations made sporadically from time to time, or from an occasional more or less isolated test item included among others of entirely different sorts.

Third, and perhaps most important, is the need for test items which reveal how well pupils really understand general principles. At present the better types of test items provide opportunity for pupils to use or apply principles in the solution of exercises or verbal problems. If the pupil is successful, and the test is valid and reliable, the teacher may infer that the pupil has learned to apply the principles sampled by the test. Only in a limited sense, however, is it valid to infer that the principle has meaning for the pupil.

An analogy may clarify the situation. There are many apparently authentic stories of dogs who have been transported far from home, and who have then succeeded in finding their way back. The dog is unable to tell how he did this. In similar circumstances an intelligent man would proceed according to certain principles, chiefly geographic in nature. He might, for example, describe his procedure as follows. "I found out I was 20 miles south of Columbus, Ohio, and I knew my home in Chicago was to the northwest. I "thumbed" a ride north into Columbus. I took a bus west to Indianapolis, and then a train northwest to Chicago." If asked *why* he proceeded in this way, he would be able to give adequate reasons.

Now a pupil may solve a problem by arithmetic, or by algebra, or write out an acceptable proof of a geometric excerise, yet become practically speechless or incoherent if asked to explain why he proceeded as he did. His knowledge of principles remains vague. He soon forgets what he does know and loses much of his operational skill at the same time.

In arithmetic the recent emphasis upon "meaning" has led some investigators to construct test items designed to measure various aspects of "meaning." Thus an example may be shown in *worked out* form, accompanied by questions about why certain steps were taken. By way of illustration, suppose the example is one of multiplication by a two-digit number. The question may be: "which of the following best explains why the 6 is placed under the 9?" In short-answer forms the pupil may select the best answer from a list of four or five given statements. It is desirable that items of types that demand knowledge of *why* something is done become as commonplace as those now used to find out if a pupil knows *how* to perform an operation.

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Similarly, for the evaluation of learning in algebra items are needed to investigate mastery of the important procedural principles. Thus if a problem is expressed in terms of two equations in two unknowns the student must clearly understand that his aim is to *eliminate* one of the unknowns. His explanation of his thinking should rise above the details of the operational methods (such as making the coefficients of one of the unknowns the same in both of two linear equations and then subtracting or adding them term by term) by which this aim is accomplished. He should recognize elimination as a general principle, to be accomplished by different methods in different circumstances. In demonstrative geometry his knowledge of logical principles should in like manner rise above the details by which proofs are exhibited. Many students will

say that "there must be a reason for each statement," but few can explain *why* in terms of the principles of "if-then thinking," recognizing "reasons" as major premises in syllogisms, and in other ways revealing their understanding of the general structure of the argument. The inclusion in tests of questions which produce direct evidence of the existence of these broader understandings should be one of the forward steps taken immediately by mathematics teachers.

Evaluation for Prediction and Guidance

One of the most important purposes of evaluation is to provide a sound basis for the guidance of pupils. It is safe to assert that the data found in school files is insufficient for good guidance of individual pupils. It is inadequate, both as to type and as to amount, when judged according to modern principles of educational theory. Limitations of space make it impossible to outline here the improvements on this situation which are being developed by the better schools. There is, however, one phase of the guidance problem which is so urgent as to require immediate attention. There seems to be growing acceptance of the "two-track" program of courses in secondary mathematics. The sorting of the pupils into at least two groups, and the prediction of their achievement in the work which they are to pursue, cannot be done successfully without the aid of good evaluation data.

Past achievement is one of the best of the available bases for the prediction of future success in similar lines of study. The improvement of achievement tests, along lines such as those suggested above, is therefore one possible way of increasing the accuracy of prediction. There is, however, some difference between the abilities which lead to success in arithmetic, algebra, and geometry. Genuine understanding of algebra requires more ability in abstract thinking than is required in arithmetic. Geometry, as taught by the best teachers, requires more ability to do deductive logical thinking than is required in algebra. Studies of the statistical correlation between test scores in these fields yield, under varying circumstances, coefficients which range between 0.40 and 0.70, and these are not high enough for accurate individual predictions. The available prognostic tests for algebra and geometry generally yield somewhat higher coefficients, but even these scores may fail to discriminate accurately in the borderline cases.

The fact is that prediction is an exceedingly complex problem. Achievement depends upon numerous variables which are difficult to control and measure. Among these are the interest and motivation, the work habits and study skills, the teaching methods and the content, that will be found in the courses for which prediction is being made. Moreover, the accuracy of prediction depends upon the final criterion of successful achievement used, and this depends upon the sort of evaluation which is used in the course. All of these factors may change between the time a prediction is attempted and the time when its accuracy can be examined in terms of what the pupil actually accomplished.

In spite of the complexities of the task, teachers of mathematics should seek to secure the kinds of data which will enable them to discover early pupils who should be advised to take courses whose chief purpose is to develop mathematical competence adequate for dealing with the affairs of daily life. At the same time teachers should determine much more carefully than is now customary the students with real mathematical interest and abilities. In order to accomplish this, teachers must first of all obtain good data on the interests and ambitions, work habits, and similar characteristics of pupils, as well as facts about their past achievement in mathematics courses and their general ability as measured by intelligence tests. Secondly, teachers must determine

more definitely than is now common just what successful achievement in algebra, geometry, and other mathematics courses should imply. Third, they must insure that this achievement is properly evaluated. Fourth, they must analyze the relationship between the data available for prediction and the results achieved by individual pupils in their local situation. Particular attention must be given to the borderline cases. By regular review of such cases the predictive criteria should be refined until errors in judgment have been reduced to what appears to be a minimum. This process is never complete. Inevitable changes in teachers, textbooks, methods, means of evaluation, the character of the community, and other factors, work against any permanent or routine procedure. Finally, careful attention should be given to the data used to make decisions as to whether or not a pupil who has been traveling on one "track" should switch to the other.

It may appear to some that the evaluation activities outlined above call for more work than they are worth. Nevertheless, if the purposes of setting up the two-track program are to achieved, students who are to take the track which emphasizes the more formal and abstract work in algebra and geometry must be carefully selected. Standards in these courses should be raised appreciably in most schools. This cannot be accomplished as long as students who are incapable of meeting high standards are permitted to enroll in the courses. The very competent and the quite incompetent cases can be readily identified. Hence attention must be focused upon the determination of the thresholds—in metaphorical terms, they should be well lighted and provided with a doorman who demands clear identification before the student is admitted.

The elevation of standards which can follow such a program should not be confined to the work in algebra and geometry. Many students now permitted to do poor work in these courses are capable of doing *good* work in general mathematics. The standards for the other track can and should be raised in many classes. Later some of these students, having demonstrated that they now have a foundation adequate for the academic type of course, may be enrolled and do acceptable work in these courses also.

There have been, and there will continue to be, many proposals for improvement in curriculum and methods. If attempts to put these proposals into effect are not carefully evaluated, there is no dependable way of appraising the success of the effort. Without steady progress in evaluation, as well as in curriculum reform, adverse criticism of mathematical instruction, by colleges and by the general public, will continue. Good evaluation will, however, provide the data necessary for the most effective answers to such criticism; namely, proof that progress is being made.

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Training of Teachers of Secondary School Mathematics

By HOWARD F. FEHR
Teachers College, Columbia University, New York 27, N. Y.

THE reports, research, and proposed curricula for the training of teachers of mathematics have been numerous within the last thirty years. Outstanding have been those by Archibald,[1] Sueltz,[2] and Turner,[3] not only for the review of the training existing at the time of their report, but also for the proposed program they offer for future training. Their proposals, along with the many other contributions over this generation of time have an almost unbelievable uniformity in the listing of advanced courses of mathematical study and the type of pedagogy desirable and necessary for successful teaching. The pattern is: much pure mathematics, including its foundations, a course or two in applied mathematics, a course in psychology, a course in methods of mathematical teaching, and collegiate study in the minor related fields of knowledge. We are apt to conclude that the thinking is much like that of the two little girls who met for the first time. Said the first girl, "my father went to Harvard." The second girl replied, "my father also went to Harvard," and as an after thought added: "Is there any other place to go?" And so each new proposal of training of teachers of mathematics seems to tacitly append "Is there any other type of training to be given?," as though we had arrived at a static ideal.

The very fact that reports continue to come forth on this important topic is indicative of some dissatisfaction of the results or some failure to achieve the program set forth. It is also indicative of the awareness on the part of the trainers, that the changing concept of the purpose of secondary school education calls for a necessary change in preparation of teachers to meet the new objectives of education, and new and broader demands on teacher services to the school. It is evident that this concept of a changing school in a changing society does not permit any final proposal of an ideal training program. Yet in all this change, there are certain fundamental principles upon which we can base our study of the type of training that seems most promising of producing the teacher of mathematics to meet the needs of secondary schools. These principles would no doubt be stated in different forms, with different stress of importance, by different groups of educators. That is a minor problem. In this article we shall list these principles as (1) What is mathematics?, (2) How did mathematics come about?, (3) What are the purposes of mathematics in the affairs of man?, (4) How does the mind come to grasp mathematics and use it? It is suggested here that the degree and extent to which a teacher actually masters these aspects of mathematical education will determine the degree and extent to which he can successfully teach mathematics. This degree should be determined at any particular time, by the requirements of the secondary school program.

It would appear axiomatic that a

[1] Archibald, Raymond Claire, "The Training of Teachers of Mathematics." *Bulletin*, 1917, No. 27, Department of the Interior, Bureau of Education.

[2] Sueltz, Ben A., "The Status of Teachers of Secondary Mathematics in the United States." Ben A. Sueltz, Cortland, New York, 1934.

[3] Turner, Ivan Stewart, "The Training of Mathematics Teachers for Secondary Schools in England and Wales in the United States." *The Fourteenth Yearbook*. National Council of Teachers of Mathematics, Bureau of Publications, Teachers College, Columbia University, New York City, 1939.

teacher must *know* the subject he is expected to teach. But to know mathematics may mean anything from a superficial knowledge of high school subjects, to a complete mastery of many of its higher branches. Eric T. Bell[4] has estimated that it would take twenty or more geniuses a life time of study to master all the modern mathematics now known. How much of this must a high school teacher know? He must know much more than he teaches, very much more. He should know what algebra is, and how it is extended in later work by those who continue the study of mathematics. He should know the foundations of algebra, so that, although he does not develop the subject in a rigorous logical fashion for his high school pupils, yet he will teach correct concepts. He should have the same type of knowledge of geometry. He should recognize an intuitive or inductive acceptance of geometric and algebraic facts and relations as contrasted with those that are proved in some deductive system. He should know analytic geometry, and trigonometry, recognizing the almost inductive system of development of these subjects in the high school and first year college courses as contrasted with some possible deductive development. He should know where and how all these subjects are further extended and used in the calculus. This of course implies that he must know the elements of differential and integral calculus. He must also know the nature of mathematical structure, the logical foundations on which the subject is built, and its particular methods of extension. And depending upon whether he is a teacher only of general non-college preparatory pupils, or if he is a teacher of scientific college preparatory pupils, he has need for mastery of these mathematical studies in varying degrees.

How shall we insure this basic knowledge of what mathematics is?

Most schools set up a prescribed and elective list of courses in pure mathematics for the prospective teacher of secondary school. But taking particular courses and passing them does not insure a knowledge of what mathematics really is. One has only to consider the thousands of students who have passed courses on the calculus and are utterly ignorant of the concepts of variable, function, derivative, and integral. They learned tricks that enabled them to get through examinations, they did not learn calculus. The same can be said of differential equations, higher algebra, modern geometry, and the other courses prescribed as necessary dosage for learning what modern mathematics is. Taking such courses may be necessary but the sufficiency has been demonstrated as missing in too many cases. Perhaps a different method of teaching collegiate and graduate mathematics to prospective teachers must be initiated, under instructors who are vitally interested in the secondary school program, and who will seek to give real understanding of basic concepts rather than mechanical and manipulative skill in the operations. If this is not done the teachers will continue to teach in high school in the same meaningless mechanical way, as they were taught in college.

A teacher should know the history and development of his subject down through the ages. No teacher of arithmetic can do justice to the subject who is ignorant of the long struggle to achieve our modern decimal place system, of the various other number forms of the older civilizations, and of the growth of the systems of measures, especially the metric. A teacher of algebra cannot appreciate the methods of solution of equations, nor show the power of modern algebraic methods, who has not had acquaintance with the rhetorical, syncopated, and symbolic stages of algebraic development. The treatment of sections of a cone by the Greeks contrasted with modern methods of analytic geometry is as indicative of the progress of human intelligence as is any other field of subject

[4] Bell, Eric T., "The Queen of the Sciences," Williams and Wilkins, 1931.

matter. The aims and desires of great mathematicians, the analysis of their mathematical methods and thinking, how they came to make mathematics their life work, and the ultimate outcome of their creations, if used judiciously, can be a most stimulating tonic for the continued study of mathematics by high school pupils. The history of mathematics is the means by which we come to understand our great heritage in this field of knowledge and without this heritage it will be difficult to pass on to subsequent generations that which is of priceless value, and to discard that which has served its purpose.

Courses in the history of mathematics can help to bring the story of the creation and growth of mathematics to the teacher. But such courses can develop into facts, chronological dates, names of men, and dissociation from real life and related fields of knowledge. Even when taught with proper spirit, they supply but little of the needs of the high school teacher. There must be wide and extensive reading in the field of historical and present day developments on the part of every individual teacher. Within the past twenty years, a number of excellent historical and cultural treatises on mathematics have appeared, a proper selection of which can give deep insight into the spirit of mathematics, its structure, and its growth. The training of a teacher does not end with his initial formal collegiate work, but as is true in any profession, it goes on throughout his entire career.

Perhaps the question asked of teachers more than any other, is "where is mathematics used?" The teacher in turn passes the question on to the training teacher or the mathematics instructor, and the answers range from mathematics for its own sake, to theoretical applications, to the use of simple formulas. And there are many who quarrel over the purposes of mathematical instruction. Unless the teacher has a rather comprehensive view of the place that mathematics occupies in the affairs of man, he will not only be unable to satisfy the pupils' questions, but he will also be unable to defend the place of mathematics in the secondary curriculum against the arguments of those who see no value to be gained in its study. The uses of mathematics, both practical and cultural must form a part of the education of the future teacher.

The real applications of elementary mathematics are relatively few and usually involve concepts in other subject matter fields. This would indicate that a teacher have some knowledge of a number of related fields such as Physics, Chemistry, and Economics, so that he may study the applications of mathematics in these fields with understanding. The teacher should also know applications of mathematics to cartography, to astronomy, to navigation to mechanics, to light and sound, to electricity, to insurance and finance, and in general in all those fields which are used in interpreting our physical and social environment. This would be in accord with the report on the Reorganization of Secondary School Mathematics published in 1923, which states the primary aim of all mathematics instruction is to give us an insight and control over our environment.

There is another practical use of mathematical training, the application of modes or patterns of thinking. A teacher must not only know the logical pattern by which geometry is developed, but where and how this pattern may be applied in the examination of other fields of knowledge. The nature of proof and its use in everyday life, simple logic, have always been in the province of mathematical training, and until there is a change in the educational program, it is designed to stay there. The syllogism, consistent and inconsistent hypotheses, contraries and contradictories, the validation or invalidation of theories, necessary and sufficient conditions, existence theorems, impossible and unsolved problems are taught in the mathematics class but are the property of all realms of thought that shape our everyday life. Whether these phases of mathematical

structure are to be taught, as Jacobi is supposed to have insisted, "For the greater glory of the human mind," or, as Fourier believed, for the investigation of our physical universe, we shall leave to the debate of the pure and applied mathematicians. The high school teacher must be well acquainted with both aspects of the values of the subject.

The training program in most training institutions is at the present inadequate in giving the practical phases of mathematics. It is again evident that courses in the applications and the structure of mathematics can supply some, but not all, the background for the teacher. The study of other related fields from the mathematical point of view is likewise a necessary step in the training. All of these studies should be taught from the professional point of view, by trainers who are not only subject matter specialists but also students of secondary education. With the study there must be outside reading and search by the trainee that does not cease at the end of the course.

Having attained a successful outcome of all the previously mentioned training, yet without a knowledge of how to teach, most persons would be lost in front of a high school class. The Doctor of Philosophy in pure mathematics cannot speak the language of high school pupils, or quickly adapt himself to their stage of mental maturity. The pure historian has a philosophical aspect and interest incomprehensible to the high school pupil. The engineer and the logician use technical complexities learned through years of study. What is needed is the "humanization" of all these phases of mathematical study, that is, the simplification and the presentation of mathematics so that the child will understand, will be interested, and motivated to further study. The teacher must be taught how to teach.

This part of the training is usually given in courses on Methods and Educational Psychology, but here again the passing of the course is no absolute criterion that the teacher will be successful. How many bright students have made straight A's in Methods and Psychology only to fail in their Practice Teaching? Perhaps the courses are too general and too much theory rather than practical and applied psychology, but a start must be made somewhere. In the short time allotted to the study of subject matter, methods and psychology, we cannot hope that all our students will learn how to "humanize" their subject. We must expect that the future teacher will continue his search of how to teach when he goes out on the job.

The spirit in which all this study should be done I have shown elsewhere.[5] The program is so large that educational leaders have long felt the need of the equivalent of an additional or fifth year of professional training. For a number of reasons it is wise to spread this study, if possible, over several years of in-service teaching. The psychology of teaching becomes vitalized by actual application in the classroom and educational theory can be put to the test as it is studied. The new applications of mathematics are not crammed and forgotten, but are developed gradually over a period of years. In this way the purposes of mathematical instruction become clearer and are fused slowly, but with real insight into the whole goal of the education of the child. There is also more time for collateral reading and broadening of experiences in the whole field of education. The primary purpose of added training is not in subject matter specialization, but in a broadening of the educational outlook in the subject matter field.

The fifth year of program could well include all three categories: pure subject matter, background courses, and courses in methods of teaching mathematics. The pure subject matter would be taught partially as an extension of the teacher's knowledge in mathematics, but greater

[5] Fehr, Howard F., "Teachers of Mathematics," THE MATHEMATICS TEACHER, Vol. XLI, Number 6, October 1948.

stress would be laid on the discovery of how his mind comes to grasp the new concepts, the new methods, the new symbols, and the difficulties that block learning. In this way he not only learns to appreciate the difficulties his own pupils may have in learning work new to them, but if he is alert, he reveals to himself a psychology of learning that can have application in his own classroom teaching. The background courses can be the most illuminating in the teacher's whole academic training. Mathematics in art, in music, in nature, in crystallography, in all forms of science, and as a supreme structure of thought, can give appreciations of beauty, of power, of mental prowess, and of the progress of civilization, that will make teaching the subject a joy forever. The methods classes can make use of all this background and subject matter experience in analyzing the problems of teaching the secondary school mathematics. The whole study can be rounded into a coherent program by a professionalized study of secondary school mathematics. Such study would investigate the elementary mathematics of arithmetic, algebra, geometry, analytics, and calculus as to its foundations, its structure, its applications, its several meanings and methods of teaching. It is the sort of thing at a less difficult level that Felix Klein[6] proposed for the mathematics teachers of the German Gymnasium.

[6] Klein, Felix, "Elementary Mathematics from an Advanced Viewpoint." Translated by Hedrick and Noble, Volumes I and II, Macmillan Company, New York.

The most important factor in the training of teachers is the trainer. For a period of four years, we have under our tutelage, an aspirant to our profession. It is our responsibility first to discover whether the student can meet the exacting standards of personality, scholarship, and professional outlook, and not to pass on those who do not. It is our privilege to bring out all the latent abilities, to develop character and personality, to inspire to service, and to promote high scholarship in every potential teacher. The courses we teach are only a means to an end and it is the guidance we give, the motivation to know that we instill, that makes for real training.

A better, more vitalized program of teacher training must always be sought for the improvement of teaching. A greater contribution lies in the continued study of teaching problems, by the teacher, throughout his entire career. This may consist in taking special courses from time to time in pure and applied mathematics or in psychology and teaching; of planning and carrying out experimentation in his own classes and reporting the results in current professional periodicals; of subscribing to and reading the professional magazines both in subject matter and in teaching; of attending institutes, workshops, and conventions regularly and with an earnest search for contributions to better teaching; and in a growing understanding of the value of mathematics in the general education of free men.

"Thou that teacheth, teach thyself."

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Keeping Ahead Professionally

By E. H. C. HILDEBRANDT
Northwestern University, Evanston, Illinois

A SHORT time ago, I inquired of my dentist about the new method for preventing some of the tooth decay in children by the application of a weak sodium fluoride solution, whether this treatment would benefit my children, and whether he administered such treatments. He replied that experiments had shown that 40 per cent of tooth decay could be eliminated by applying this solution and that if I wished, he would be willing to apply this treatment to my children.

This experience has its moral, of course. We expect our dentists to know about the latest developments in their field, we expect our doctors to know of the latest medical treatments, and we expect our merchants to carry the latest stock of merchandise. And in turn, is it not expected of us, as mathematics teachers, to keep ahead professionally?

There is good evidence to support the statement that teachers of mathematics are vitally interested in keeping ahead professionally. One reason for this statement can be illustrated by the following occurrence late in 1947.

An all-day conference for secondary school teachers of mathematics was announced for the second Saturday before Christmas. Invitations were sent to schools within a 100-mile radius of Chicago. One can expect almost a 100-per cent attendance if a teachers convention is held on a school day and schools are dismissed to encourage attendance. But will teachers be willing to give a day of their own, particularly such a day as this one, the second last Saturday for shopping before Christmas, just at the time when there are so many things that need to be done before the holidays? In many cases, persons planning to attend would have to leave much earlier this Saturday morning than they did for their daily arrival at school at eight. Then there was the weather! For a week or more preceding this particular Saturday, the weather was very cold—there had been enough snow to make driving treacherous—the kind of driving weather that makes radio announcers suggest that one keep one's car at home. Prior to the date of the meeting 260 teachers sent in reservations. Would you expect all 260 to be there? The day of that conference there were 272 mathematics teachers present! The reason for such attendance rests in the fact that mathematics teachers are interested in professional growth.

There are various ways in which such growth can take place. Among these are the following:

(1) taking summer and extension courses in mathematics, in the teaching of mathematics and in related fields, in our colleges and universities;

(2) taking workshop courses for shorter or longer periods during the summer;

(3) attending summer institutes on mathematics teaching;

(4) attending one- or two-day conferences and conventions held during the school year;

(5) joining organizations for mathematics teachers which function in a local area, in the state or nationally;

(6) reading the latest articles in the official journals of these organizations as well as reports of recent studies in yearbooks and other publications.

Yet one can do all these and more, and not keep ahead professionally. There is one more vital factor which must not be overlooked. It can best be illustrated by an oft-told story:

A certain retired clergyman was invited one Sunday morning to preach at a little church in a town about twenty miles from his home. On that particular morning he took his little six-year-old grandson along with him and when they arrived at the church they found there was still sufficient time before the service was scheduled to

begin. So they made an inspection trip of the church, visiting the Sunday-School rooms, the main church auditorium, and the balcony. In passing through the back part of the auditorium itself, they chanced to pass a box into which it was customary to drop some coins for special purposes. The clergyman reached into his pocket and put in a few coins. The church service was not too well attended but it was finally over and the little boy looked forward to returning home. Following the service, one of the deacons came up to the minister and said: "Pastor, you preached a fine sermon this morning! We wish we could pay you for your help but unfortunately our church treasury is empty and the best we can do is give you whatever is in the box which is in the rear of the church auditorium." So saying, he went to the box, opened it, and gave the minister the same coins which the latter had placed in it earlier that morning. The minister graciously thanked the deacon while the little boy looked on greatly puzzled. On the return trip home, the little boy was quiet for a long time—an unusual thing for this small boy. Finally, he gave out a deep sigh and said "Grandfather, you know, if you had put more into that box, you would have gotten more out of it."

As mathematics teachers, we too find that we get out of the summer and extension courses, the workshops and summer institutes, the conventions and meetings, and the journals what we put into them. We may listen to the best lecturers in our courses and institutes presenting some of the finest suggestions for our teaching, but if we do not take part in the opportunities offered for discussion or rethink them in terms of our own teaching situations, "It profiteth me nothing." In our workshop courses, we again get out of them what we put into them. If experiences and data are carefully collected, classified, analyzed and then submitted in a well-planned report, then the benefits in one's teaching will surely be felt. It is important indeed that every mathematics teacher join a local or sectional organization of mathematics teachers as well as the state mathematics section of the state teachers association and the state council of teachers of mathematics and that every teacher of mathematics in this country should be a member of the National Council of Teachers of Mathematics. But membership in these organizations can be of a passive nature or it can be an important means to continued growth. We need to take part in the various phases of planning of local and other meetings, we need to invite others to attend these meetings and conferences, we need to take an active part in the discussions and to contribute suggestions and solutions from our own experiences. Not only should we have the opportunities to listen to suggestions which relate to improved mathematics teaching but we should also have opportunities to personally take part in discussing these questions and then apply them in our teaching.

Continued participation in the various mathematics activities will lead to growth in our profession and this growth must never stop. Even the little six-year-old boy could figure out that his grandfather could have taken more out of the collection box if he had put more into it!

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Make your reservations *now* for the Annual Meeting of The National Council of Teachers of Mathematics at the *Lord Baltimore Hotel in Baltimore, Md.*, March 30, 31 and April 1 and 2, 1949.

Planning a Secondary Mathematics Curriculum to Meet the Needs of All Students

By Dale Carpenter

Supervisor Mathematics Education Section, Los Angeles City School Districts, Los Angeles, California

The title of this article infers that the present-day mathematical needs of secondary students are not adequately met by the present-day mathematics curriculum. A brief, historical review of the secondary mathematics curriculum should be of help in determining the validity of the preceding assumption.

I. The Position of Algebra and Plane Geometry in the Mathematics Curriculum

The problem of reorganization or reconstruction of the high school mathematics program has existed almost from the time of the inception of the high school.

The traditional "beginning algebra" or "ninth year algebra" and plane geometry were offered in colleges at the time the high school was first established. These courses were moved to the high school level soon after the high school had become an accepted part of the educational scene in this country. The reasons for the change may have been because high school teachers desired status for themselves and their institution and also because the college teachers wished to free their curriculum from beginning algebra and geometry in order to make room for more important things.

This downward trend of mathematics placement has continued until recently and has extended to the junior high school.

The junior high schools were organized to act in the capacity of a "buffer" state of armed neutrality between the carefree child of the elementary school and that blasé individual, the senior high school student. However, the conception of mathematics as "preparation for mathematics" was so firmly established that the junior high school lost its role of watchful waiting and was taken over by the academic wolves of "preparation." Not only did algebra become standard fare for the ninth grade student in the junior high school but also, in many cases, for the eighth grade student and, in a restricted sense, for the seventh grade student.

The arguments substantiating this continued downward trend were simple and to the point. Perhaps you have heard some of them:

1. If algebra is good for ninth grade students why not give it in the eighth grade also? It won't do them any harm.
2. I can teach algebra to any one. It's all in knowing how!
3. Algebraic techniques are instruments of power. We are morally obligated to put these time-saving devices and instruments for thinking into the hands of students as soon as possible.
4. If algebra is introduced in the seventh grade and continued in the eighth grade, the students' ninth grade achievement in algebra will be greater than if they had had no previous introduction to the subject.
5. If algebra is presented to eighth grade students they will become interested and stimulated by the new approach to mathematics.

However, a reaction to the downward trend of mathematics content set in about twenty or twenty-five years ago. This reaction became so strong that algebra was eliminated entirely from the ninth grade in many schools. Some of the factors which influenced this reaction were:

1. A greater percentage of students of junior high school age remained in school.
2. More time was being given to the fine and practical arts.
3. Many administrators believed in an educational philosophy which placed the social studies as the center of the pupil's activities and considered mathematics as a necessary evil, to be learned as a "tool" when and if it was needed.

As the ratio of high school population to total population increased, so also did the instructional and learning difficulties in algebra and geometry. These difficulties became so great that instructors looked for measures which would alleviate the situation.

Ability grouping was tried and instruction adapted to the individual. "Algebra Z groups" and even "Geometry Z groups" were formed. Algebra and geometry were defended on the grounds of cultural necessity, democratic privilege, improvement in the ability to think, and mental discipline. The idea that geometry might possibly improve one's ability to solve practical problems was frequently omitted as an objective of instruction in that course. The idea seemed to be to make sure that no secondary student should be denied the privilege of memorizing the formula for solving quadratic equations and the privilege of memorizing a formal proof and reaching those magic letters, Q. E. D., without having the teacher say, "Yes, it was to be proved, but is it proved?"

In spite of such adaptations as were made in algebra and geometry these subjects lost ground as a part of general education and they remain a part of the curriculum as a college entrance requirement and as the basis of the mathematics requirements for science and engineering majors.

II. Mathematical Needs of Secondary Students

The major purposes and objectives of mathematics instruction must be kept in mind when planning curriculum content, curriculum materials, and when planning daily instruction. *The Guidance Pamphlet in Mathematics*, prepared by the Post-War Commission on Mathematics, gives us a practical break-down of the mathematics needed in four major areas. These areas are listed as follows:

1. Mathematics for Personal Use
2. Mathematics for Skilled and Semi-Skilled Workers
3. Mathematics for College Preparation
4. Mathematics for Professional Workers

In order to plan an effective mathematics curriculum we must keep in mind the nature of the secondary school population as well as the specific mathematical objectives which must be met by such a curriculum. The following information will assist us in understanding the nature and the mathematical needs of secondary students.

One hundred per cent of the high school graduates are faced with the personal problems of living. Approximately fifteen per cent of the high school graduates enter college. Of this group less than half receive four or more years of college work. Less than 6.5 per cent of the labor force in the United States are classified as professional, whereas approximately sixty per cent are industrial workers, skilled and unskilled, and approximately thirty per cent are in commercial occupations. Also we must remember that approximately forty per cent of the students who enter the ninth grade do not graduate from high school. When considering any secondary curriculum problem we cannot dismiss these drop-outs or quitters as unimportant. Many of them leave high school because of lack of interest and failure to see that the high school has anything valuable to offer to them.

A majority of the college preparatory students enter professions which do not require a specialized mathematics background. However, the high school has solved the secondary mathematics curriculum problem for college preparatory students who will need a specialized mathematics background, and for those who will not need this background, very neatly. Students in both groups generally take algebra and geometry in the ninth and tenth grades and those who need the specialized mathematics background continue with mathematics in the eleventh and twelfth grades.

The assumption apparently has been made that the mathematics education

needs of the general college preparatory student are met by courses in algebra and geometry, that these same courses are also the best courses for the beginning work of the mathematics and science majors, and that for the general college preparatory student, the mathematics ability acquired in the ninth and tenth grades in algebra and geometry will take care of his mathematical needs in the eleventh and twelfth grades and in college. An analysis of the mathematical needs of the general college preparatory student and of the mathematics and science majors should be made and suitable programs should be prepared to meet the specific needs of each of these groups.

The mathematical needs of those students who enter industry are extremely varied. Included in this group of students are those who enter trades which may require a high degree of technical skill but very little ability in related mathematics. However, some of these students prepare for trades which require considerable knowledge of related mathematics as well as a high degree of technical skill. Such trades as mechanic and die-maker are illustrative of this type.

Schools have endeavored to meet the mathematical needs of the industrial group by such courses as shop mathematics, related mathematics and industrial mathematics. Due to the tremendous range of the mathematical abilities of the students in such classes and to the fact that such courses have been generally considered to be "dumb-bell" courses, they have usually failed to achieve, to any extent, their purpose.

The mathematical needs of those who enter the commercial field are probably not as varied as those of the industrial group. The high school has tried to meet the needs of this group with courses in business arithmetic and the related arithmetic which is inherent in commercial and bookkeeping courses. The success of courses in business arithmetic has also been very "spotty." This lack of consistent results has probably been due to the same reasons which have prevented the courses for students who enter industry from becoming successful.

The senior high schools have also offered courses in remedial arithmetic, review arithmetic, senior mathematics, and refresher mathematics. The purpose of the courses in remedial and review arithmetic seems obvious. The purpose of senior mathematics has generally been to prepare students for the quantitative consumer problems of everyday living. Refresher mathematics was a war baby. Its avowed purpose was to refresh or restore certain mathematical abilities which had been forgotten, due to neglect during the upper grades in high school.

III. Problems of Organizing a Complete Mathematics Curriculum

The greatest difficulty which prevents us from planning and establishing a mathematics curriculum suitable for all secondary students is the firm conviction on the part of many teachers that exposure to the wonders of algebra and geometry creates some miraculous power in the individual and places him on an entirely different cultural plane than he was capable of enjoying previous to his indoctrination. In fact, the belief is quite prevalent that a year's study of algebra and a year's study of plane geometry acturally transform a student into an entirely different type of individual. Through the mystic power of quadratic equations and formal proof an ordinary human being is transformed into a member of that elite group known (for want of a better definition) as "college material."

Another serious difficulty which prevents any real reoganization in mathematics curriculum from taking place, is our failure to recognize worthwhile objectives of mathematics instruction other than satisfying a college entrance requirement or as a prerequisite for more mathematics. This narrow conception of the

objectives of mathematics instruction has frequently resulted in a static point of view which may be described by the following statements which are often used to defend the status quo of the mathematics program:

1. Mathematics is the perfect science. It must always be developed in logical sequence.
2. Each part must be "mastered" in order to proceed to the next portion.
3. The teaching sequence is determined almost entirely by tradition. New applications of mathematics are further proof of the success of the traditional mathematics program.
4. The reasons for the inability on the part of the student to master a given phase of mathematics is easily ascertained. The former teacher simply failed to instruct properly or the student has reached his mathematical saturation point.
5. The prescription: arithmetic until mastered, then algebra, plane geometry, more algebra, trigonometry, and solid geometry, in the order given.
6. The principal objective of mathematics instruction is to develop reasoning power.

Following are several additional reasons why progress in establishing a more complete mathematics program has been slow:

1. Teachers prefer to teach the traditional mathematics courses due to their training and lack of practical experience.
2. Students and parents believe that enrollment in the traditional courses is a mark of distinction.
3. Lack of agreement upon content, sequence, and materials based upon vocational needs of all groups of students.
4. Many mathematics teachers do not have adequate mathematics background. Many secondary mathematics courses are taught by teachers who have neither interest nor training in mathematics.
5. Mathematics teachers frequently have not realized the necessity of different types of secondary mathematics programs. Many teachers in the senior high school have looked upon senior high school arithmetic as a necessary evil, not quite worthy of association with the more respectable members of the family. It was to be condoned because of the evident failure of the elementary and junior high schools to do their job and because many students could be salvaged for algebra by means of arithmetic classes.
6. Administrative difficulties, such as lack of teacher time and crowded schedules of students.

Any program of curriculum development which attempts to establish a curriculum which will meet the basic needs, as defined by the Commission on Post-War Plans of the National Council, must be planned with the cooperation of all groups affected by the program. These groups are, (1) junior and senior high school mathematics teachers, (2) college mathematics teachers, (3) teacher training directors, (4) junior and senior high school counselors, (5) junior and senior high school administrative officers. These groups must reach a common understanding of the nature of the high school population and an agreement concerning the mathematical needs of all secondary students. It is only through such an understanding and agreement that a modernized mathematics curriculum can be planned and put into operation.

IV. Planning a "Complete" Program of Mathematics for the Los Angeles High Schools

Some knowledge of the economic environs of Los Angeles and of the Los Angeles School System is essential to the understanding of the educational needs of the students in the high schools of the district.

The Los Angeles high school district includes all types of residential communities and a variety of industries. Some of the major industries of the Los Angeles area are, (1) steel, (2) automobile tires, (3) automobile assembly, (4) aircraft, (5) petroleum, (6) motion picture, (7) garment.

There are 32 junior high schools, 34 senior high schools, and 11 six-year high schools in the Los Angeles school district. The junior and senior high school pupil enrollment is approximately 115,000.

The secondary mathematics program previous to World War II was essentially as follows:

1. Approximately one-third of the ninth grade students were enrolled in algebra. A selective process was used to assist counselors and teachers in assigning students to algebra classes.
2. Students who did not take algebra in the

ninth grade were required to take one semester of elementary business arithmetic.
3. The usual sequence of plane geometry, second year algebra, plane trigonometry and solid geometry was offered in the senior high schools.
4. First year algebra was also offered in the senior high schools.
5. Courses in shop mathematics, related mathematics, applied mathematics, refresher mathematics, business mathematics, senior mathematics, and industrial mathematics were offered in various high schools.
6. An arithmetic test was given to all A-11 students, other than mathematics majors, and those who failed to achieve an eighth grade ability were required to take at least ten weeks of arithmetic.

There is no mathematics requirement for graduation from high school in California, which accounts for the fact that approximately fifty per cent of the high school students in Los Angeles graduated without having had any mathematics courses in the senior high school. Approximately twenty-five per cent of the graduates had completed one year of algebra and approximately twenty per cent had completed one year of algebra and one year of geometry. Approximately six per cent had completed three years of high school mathematics and less than half that number had completed four years of high school mathematics.

Shortly before the beginning of World War II, the need for the reorganization of the non college preparatory mathematics had become apparent. These courses had no sequence of content and very little coordination as regards to purpose, guidance, and content.

A committee of senior high school mathematics department chairmen and teachers was formed for the purpose of studying the problem of reorganizing the non college preparatory mathematics and for the purpose of formulating an improved sequence of courses in this field.

Progress of the committee was slow due to differences of their own experiences, to differences of curriculum planning in various schools, and differences of interests and abilities of the students for which the courses were being planned. However, a thorough study of current curriculum practices was made and the advice of interested commercial and industrial leaders was sought.

The members of the committee found that they needed to come to an agreement concerning the major purposes of non college preparatory mathematics and certain basic principles which would determine the nature and manner of operation of the courses. These principles are summarized as follows:

1. Mathematics courses suitable to the needs and abilities of all high school students should be offered.
2. Since an administrative limit exists as to the number of different types of courses which can be offered, and since a very careful analysis of the mathematics requirements of the home, of business, and of industry revealed a decided similarity of mathematical content, suitable courses would be arranged on the basis of mathematical content, and that emphasis would be placed upon understanding of elementary principles and upon applications and problem solutions.
3. A high degree of flexibility in making pupil adjustments between courses should be maintained at all times.
4. Suitable materials would be selected from a variety of textbooks in order that the committee could make recommendations of content based upon need and adaptability to student abilities.

After reaching agreement upon the principles of operation for the non college preparatory mathematics, the committee planned a sequence of four Basic Mathematics courses. This *Basic Mathematics Program* was subsequently put into operation in the Los Angeles high schools.

This Basic Mathematics program is described as follows:

1. Each course is a one semester course.
2. A mathematics Guidance Test is used to assist in placing students in appropriate courses.
3. *Basic Mathematics I* is a remedial course in very simple arithmetic for very slow students.
4. *Basic Mathematics II* is a substantial review course in arithmetic. Students of poor arithmetic ability who desire to elect any of the more advanced courses in

basic mathematics, industrial mathematics, or algebra are counseled to enroll in Basic Mathematics II. A-11 students who fail to achieve the minimum arithmetic requirement are enrolled in Basic Mathematics II.

5. *Basic Mathematics III* is a course in mensuration, including the use of appropriate formulas and a variety of practical applications.
6. *Basic Mathematics IV* is a course in consumer mathematics, including emphasis upon elementary statistics. A good background of arithmetic is required for enrollment in either Basic Mathematics III or Basic Mathematics IV.

Enrollment in any Basic Mathematics course is determined by interest, past achievement, and ability as shown by the Guidance Test. None of these courses is a prerequisite to any other course.

Students may transfer between Basic Mathematics classes, between Basic Mathematics classes and algebra classes, and between Basic Mathematics classes and industrial mathematics classes at convenient times within the semester. These transfers are dependent upon interest and demonstrated ability.

Guidance in the field of mathematics has definitely improved since the foregoing plan has been in effect. Both counselors and teachers assist students to make a choice based upon ability and interest. In cases of misplacement, adjustments can be made. The Basic Mathematics and the industrial mathematics have achieved status. Achievement in the traditional courses has improved, due to better guidance and placement of students, and in many schools enrollments in algebra have increased rather than decreased.

During World War II classes in industrial mathematics were started in some of the senior high schools. These classes were planned for boys who expected to go directly into industry from senior high school and for boys who expected to enter military service soon after graduation. The work done in these classes proved very satisfactory.

Soon after the establishment of the Basic Mathematics courses, an Industrial Mathematics sequence of six semester courses was planned. These courses are now given in approximately one half of the senior high schools of Los Angeles. A brief description of the content and purposes of these courses follows:

1. *Industrial Mathematics I.* Arithmetic review, measurements, simple equations, evaluation of formulas, practical applications to shop work. Open to students who have credit in Algebra I or have a good record in junior high school mathematics.
2. *Industrial Mathematics II.* Fundamentals of algebra, graphs, simple mechanical formulas, industrial applications, use of measuring instruments. This course meets the mathematical needs of most Class C operators in shops. Open to students who have credit in Industrial Mathematics I or Algebra II.
3. *Industrial Mathematics III.* Review of the fundamentals of algebra, factoring and quadratic equations, applied geometry, solution of the right triangle by trigonometric functions, industrial applications. This course meets the mathematical needs of most Class B operators in shops. Open to students with credit in Industrial Mathematics II or in Geometry II.
4. *Industrial Mathematics IV.* Simultaneous equations, roots and powers, logarithms, trigonometric solution of oblique triangles, mechanical formulas, work, power, and energy, elementary strength of materials. This course meets the mathematical needs of most Class A operators in shops. Open to those students who have credit in Industrial Mathematics III or Geometry II and Algebra III.
5. *Industrial Mathematics V.* Review of algebra and plane trigonometry, mechanical formulas, compound angles, slide rule, industrial applications. Open to students who have credit in Industrial Mathematics IV or Trigonometry and Algebra III.
6. *Industrial Mathematics VI.* Solid geometry (with omission of proofs), application of trigonometry to solid mensuration, graphs of functions, applications to engineering problems involving force, work, power, and energy. Open to students who have credit in Industrial Mathematics V. Students who complete Industrial Mathematics VI have sufficient mathematical background for most technical courses such as drafting, tool design, etc.

The Industrial Mathematics program was developed with the cooperation and help of teachers and representatives from various major industries of Los Angeles. It fills a very definite need and is a respec-

table member of the mathematics curriculum. These boys move at a slower rate than the more academic type student. However, they achieve an excellent understanding of industrial mathematics.

It is current practice in the Los Angeles School system to offer algebra in the ninth grade and plane geometry in the tenth grade for all college preparatory students. Some of these students defer first year algebra to the tenth grade and plane geometry to the eleventh grade. A second year of algebra, plane trigonometry and solid geometry are offered in all high schools for mathematics and science majors.

A group of Southern California teachers and representatives from the University of California at Los Angeles have cooperated in planning a four-semester mathematics course which will provide a more practical and suitable mathematical education than the traditional algebra and geometry provides for college preparatory students other than mathematics, science, and pre-engineering majors. This four-semester course has been approved by the University of California on an experimental basis and is now in operation in fourteen Southern California high schools. Nineteen other universities are also cooperating and have signified their willingness to participate in the experiment.

The group, or committee, which planned this experimental program established suitable controls for the experiment and agreed upon the objectives and special emphases for the courses.* These controls, objectives, and emphases, will serve as a guide in understanding the nature of the experimental program.

Controls

1. The two-year mathematics sequence will be called *Mathematics I, II, III, and IV.*
2. The minimum length of time for the experiment should be six years.
3. The mathematical content shall be subject to revision during the six year experimental period in light of experience as the program develops.
4. A comprehensive achievement test for each of the four semesters and an achievement test for the entire four-semester course shall be prepared by the committee.
5. These achievement tests will also be given to students in the experimental schools who are enrolled in the standard first year algebra and plane geometry courses.
6. The experiment shall be confined to the Southern California area in order not to complicate the study.
7. At least five schools from the Los Angeles City School System and five Southern California schools outside of the Los Angeles City School System should participate in the program. It was the desire that a cross section of schools be obtained, including private schools.
8. The Office of Relations with Schools, University of California at Los Angeles, seems to be the logical place where the actual collection of data and compilations could be made.
9. The results of the study should be based on the records of students in the experimental group who have completed the two-year sequence and of students in the control group who have completed one year of formal algebra and one year of formal geometry.
10. The collegiate records of these two groups shall be collected at the end of the first semester and again at the time when the students are ready to enter upper division work.
11. The Office of Relations with Schools will collect not only the University of California records, but the collegiate records of the experimental students who may have gone to certain of the four-year or public junior colleges in California—these records to be evaluated in the same manner as the supplementary collegiate records. The colleges would need to be informed of the plan and their cooperation sought.
12. It is understood that the students who have gone through the two-year experimental mathematics sequence, which will include both algebraic and geometric functions and principles, will not be held for any more mathematics in the University unless their particular courses of study require it.

Objectives

1. The ability to solve practical problems within the content fields of the course.
2. The ability to use the analytical method of problem solving.

* Persons interested in this program may secure a more detailed description of the content by writing to Mr. Charles Fabing, Mathematics Education Section, Curriculum Division, Los Angeles City Schools, Room 314, 1205 West Pico Blvd., Los Angeles, California.

3. The ability to make mental calculations and to make reasonable estimates.
4. An understanding of essential mathematical vocabulary and concepts.
5. The habit of forming judgments based upon objective information.
6. The habit of doing all written work in neat, acceptable form.
7. The habit of accuracy and of checking all computation.

Special Emphases

1. Arithmetic should be reviewed as indicated in the outline and the concepts of fractions, percentage, and approximate computation will be stressed and applied throughout the course.
2. Elementary concepts of algebra and geometry will be introduced early in the course and expanded and applied throughout the course.
3. Ratio and proportion, and graphs will be expanded and applied throughout the course.
4. The application of mathematical principles to practical problems will be stressed at all times.
5. Approximate mental solutions will be stressed at all times.
6. The fundamental algebraic processes, including quadratic equations will be presented.
7. Fundamental notions of geometry, including similarity and congruency are presented informally and used early in the course. The idea of formal proof is to be stressed in Mathematics IV.
8. Mathematics will be treated as a deductive science at all times.

Full information concerning the College Preparatory Mathematics Experimental Program was given to all high schools in Southern California. Fifteen of these high schools then requested permission to participate in the experiment.

The teachers who planned to teach the experimental course and the counselors of the participating schools met and discussed the problems relating to the new program. The counselors and principals of the junior high schools which send students to the participating senior high schools, met and discussed the problems of guidance and pupil and parent cooperation. Letters were then sent to the parents of the ninth grade students informing them of the experimental plan. As a result of the complete cooperation of all groups concerned eighteen classes in Mathematics I, with a total enrollment of five hundred nineteen students, have been started without misunderstanding or difficulty. Each semester new classes in Mathematics I will be formed in each of the experimental schools.

In addition to the curriculum outcomes which may result from the experiment, increased understanding of mutual problems and improved relations between college instructors and high school mathematics teachers are already apparent. Both groups will wait for the results of the experiment with keen interest and will cooperate in planning a program that will be in accord with the results of the experiment.

Summary

The curriculum planning described in the preceding pages represents an attempt on the part of mathematics teachers in the Los Angeles high schools to provide a multiple track plan which will meet the mathematical needs of all senior high school students. This plan may or may not be suited to the needs of other school districts.

The outstanding features of the Los Angeles mathematics curriculum planning are:

1. Democratic participation and cooperation of all interested groups.
2. Agreement by the committee members concerning the objectives of mathematics instruction.
3. The planning of specific courses on a semester basis designed in accordance with those objectives.
4. Complete flexibility of the program within the administrative organization. This flexibility has helped to increase the enrollment in mathematics classes, decrease the number of failures, and improve guidance in the area of mathematics.

It is the opinion of the writer that these four features, or guiding principles, are essential in planning a complete mathematics curriculum in any type of community and that their use will result in an improved mathematics program.

The Teaching and Learning Processes in Mathematics

By WILLIAM BETZ
Rochester, New York

A STUDY of the reform movement in mathematics shows that during the first thirty years of its existence—roughly from 1893 to 1923—the reconstruction of mathematics was controlled mainly by *mathematical* leaders. Since then, the "educators" and administrators have increasingly dominated the scene. It appears that each of these groups has had a weighty share in bringing on the present crisis. Briefly, the mathematicians tried to build an imposing mathematical edifice, but only too often forgot to take into account the *educational* problems of the school. The educationists fell into the opposite error, that of glorifying the pupil but proceeding to forget mathematics. Hence it should be perfectly obvious that during the period that lies ahead of us we must correct this dual blunder by devising a program which shall give due recognition both to mathematics and to the learner.

We are here urging a re-examination of the teaching and learning processes in mathematics. And because of the vast scope of this subject, we shall stress mainly such phases of the pedagogy of mathematics as are still in need of greater emphasis or of further discussion, clarification, and redirection.

1. ORGANIZATION VERSUS INCIDENTAL LEARNING

Mathematics has been called the "science of order." Its very existence depends on order. Hence we are obliged to ask how such a body of ordered knowledge can "emerge" when nothing is "fixed in advance," or when we are to depend solely on "novelly developing situations."

This plan of "incidental learning," long favored by numerous "progressive educators," is certainly contrary to both individual and racial experience. In every walk of life we soon find out that disorder is the greatest enemy of efficiency and hence of success. Why should it be surprising that in the mental realm, likewise, order is of paramount importance? We all know that random impressions, as a rule, fade from our memory, and that it requires conscious effort to "get hold" of things we wish to retain.

Recent investigations confirm the verdict of everyday life concerning the crucial role of organization in the learning process. Thus, in the field of primary arithmetic we now have a pertinent study, undertaken by a group of teachers under the direction of Dr. Paul Hanna. Their carefully evaluated records prove overwhelmingly the appalling inefficiency of an unorganized approach.[1] At a higher grade level, such an investigation would have been even more damaging to the claims of incidental learning. Of equal but more general significance is a series of experiments in learning conducted by Dr. George Katona. His findings leave no room for doubt that organization plays a vital role in remembering.

For example, let the reader look at a set of figures like the following: 791214171922. Then let him try to recall this sequence of numbers, say an hour later. He will probably find it hard to do this. But when he is told at the outset that these figures were organized as suggested in the diagram at

[1] "Opportunities for the Use of Arithmetic in an Activity Program," by Paul R. Hanna and others, in *The Tenth Yearbook of the National Council of Teachers of Mathematics*, pp. 85–120, especially p. 119.

the right, the task of recalling this long number is quickly simplified.[2]

When the mathematical skills are to be acquired "incidentally," the consequences are always disastrous. Very soon, it becomes necessary to resort to "supplementary drill units," to "remedial lessons," and the like. That is, the very purpose of the "life-situation approach," that of providing mathematical training in a "natural setting," quickly becomes inoperative and the skills are then taught mechanically and without motivation. This difficulty has been freely admitted by "progressive" writers. Thus, Professor Melvin says:

> The place of drill in the modern school has been much misunderstood. Some proponents of change have gone so far as to teach that children can learn the number processes incidentally, completely obviating the necessity of drill. Some say that selling stamps in a post office or making change in a school store provides sufficient training so that no drill is needed. Theoretically it may be granted that there are bright children who could learn number thus, on the side—this provided their teachers could bear teaching it by interrupting and cross-cutting more important experiences to dwell upon number. *But schools which have experimented with such attempts have always failed to meet the needs of all the children without drill.* Misunderstanding is further due to failure to realize that drill is a normal part of more extended activities. It is a special activity needed in the support of other activities.[3]

All this leads up to a central question. Are there definite, trustworthy procedures or reliable psychological principles which teachers of mathematics may confidently employ in their work? In other words, is there a dependable psychology of learning?

2. Which Psychology of Learning?

Time was when we were treated to an annual crop of psychologies, after the manner of the spring season's new hats. The battle between our conflicting psychologies of learning is reflected in a number of interesting publications.[4] Which of these psychologies should a teacher of mathematics regard as authoritative or desirable?

The primary truth is that psychology is not a science in the sense of physics or chemistry. The element of prediction, and hence of certainty, is lacking. The old dualisms of matter and mind, body and soul, have not disappeared. A leading mathematician and scientific thinker tells us that

> We must await the further development of science, perhaps for centuries, perhaps for thousands of years, before we can design a true and detailed picture of the interwoven texture of Matter, Life, and Soul.[5]

This being the case, all schools of psychology are operating on distinctly hypothetical foundations. That seems unavoidable. But a fatal situation arises when a system employing numerous undefined or nebulous terms, and resting on conflicting or very debatable assumptions or beliefs, tries to take on the status of a dogma and of demonstrated certainty. Such a structure can be regarded as "science" only in a provisional sense.

That applies, in particular, to the mechanistic psychology of connectionism, which is still exerting a totally unwarranted influence in many a classroom. It is the only psychology to which the average teacher has been exposed. Untouched by persistent criticism, the mechanists continue to display their unproved formulas. "Repetition" and "reward" are still described as the prime movers which account for the whole symphony of "man and all his works."

However, nothing whatever compels us

[2] Katona, G., *Organizing and Memorizing*, Columbia University Press, 1940, p. 189. Summaries of Katona's work may also be found in Gates, Arthur I. (and others), *Educational Psychology*, The Macmillan Company, 1942, pp. 379 ff., and in Hartmann, George W., *Educational Psychology*, American Book Company, 1941, pp. 287–290.

[3] Melvin, A. Gordon, *Method for New Schools*, The John Day Company, 1941, pp. 62 ff.

[4] See, for example, "The Psychology of Learning," *Forty-First Yearbook of the National Society for the Study of Education*, Part II, 1942.

[5] Weyl, Hermann, *The Open World*, Yale University Press, 1932, p. 55.

to subscribe to these interpretations. On the contrary, the latest atomic and biological research has virtually eliminated mechanism as an explanatory principle. More than that, not a few of the world's leading scientists and philosophers have openly espoused a non-mechanistic orientation.[6] "We have left behind us," said one of them "the *childishness* of nineteenth century materialism."

In like manner, for reasons stated below, we must get rid of purely mechanistic procedures in the teaching of mathematics. Teachers of mathematics may feel quite safe in following psychological doctrines which assign a *genuine* role to such concepts as consciousness, intelligence, mind, insight, understanding, and so on. An emancipation from mechanism restores dignity to teaching, and is certainly far more in keeping with the total scheme of reality in which the drama of human existence is being enacted.[7]

3. The Teaching of Concepts

"The entire field of mathematics is conceptual," we are told by Professor Wheeler. He goes on to say,

> A person uses concepts whenever he adds, divides, multiplies, and subtracts, in fact whenever he employs any symbol . . . . A statement with regard to any kind of uniformity or generality is the expression of a concept that makes possible the rational prediction of future events. Moreover, definitions of all sorts are systems of concepts. . . . *The use of concepts and the making of generalizations are the outstanding features of intelligent behavior* from the standpoint of self-observation.[8]

How, then, shall mathematical concepts be taught?

We are not concerned here with the extremely complex theories as to the nature and origin of the basic objects of mathematics.[9] Instead, we are thinking of the several hundred terms which together constitute the vocabulary of elementary mathematics.

In former days the job of teaching concepts caused no headaches, except to the pupil. Following the textbook, the teacher calmly *assigned* entire pages of definitions, to be memorized once for all. What really happened to Johnny when he was thus catapulated into a strange world is now reflected in savage reminiscences of the older generation. "No head for mathematics," "another moron," "runs in the family," were among the caustic comments of the teacher. Much of the present unenviable reputation of mathematics must be attributed to unintelligent teaching, and in particular to the permanent darkness that was so often cast on the mathematical scene in the very first lessons.

Today we hear much about the "meaning theory." We agree that a persistent attempt should be made to establish "meanings." But here, again, we run into the same old difficulties. The psychology of concept-formation is still beset by much vagueness or by erroneous ideas which must be eliminated. The mechanists, running true to form, have long depended on *analysis* and *dissociation* for the derivation of concepts. It is merely necessary, they say, to confront the learner with a large number of different situations, all containing the common element to be abstracted.[10] For example, "by presenting

[6] See, for example, Compton, Arthur H., *The Human Meaning of Science*, Chapel Hill, 1940, Chapter II, "Human Freedom and Physical Law"; Jeans, Sir James, *Physics and Philosophy*, The Macmillan Company, 1943, pp. 195 ff.

[7] See, Hocking, William Ernest (and others), *Preface to Philosophy*, The Macmillan Company, 1947, pp. 413 ff.; Eddington, A. S., *The Nature of the Physical World*, The Macmillan Company, 1929, pp. 276 ff.

[8] Wheeler, Raymond Holder, *The Science of Psychology*, Thomas Y. Crowell Company, 1929, p. 146.

[9] See the introductory chapters of Katsoff, Louis O., *A Philosophy of Mathematics*, The Iowa State College Press, 1948.

[10] For an exposition of this doctrine, see Gates, A. I., *Psychology for Students of Education*, The Macmillan Company, 1922, p. 307; Thorndike, Edward L., *Educational Psychology*, New York, 1919, pp. 159 ff.; Thorndike, Edward L., *Human Learning*, The Century Co., 1931, pp. 139 ff. In arithmetic, the doctrine was innocently used by Pestalozzi and his followers. It has been rejected by Dewey, John (*How We Think*, D. C. Heath and Co., 1910, p. 52, and 1933 revision, pp. 155 ff.); by Husserl (1891);

the quality of 'square' in various contexts —boxes, tables, pieces of paper . . . —, the association of squareness with other qualities is weakened until squareness finally drops out of these contexts altogether, as a ripe apple drops from a tree."

We agree with Professor Bode's admirable critique of this theory which, on close examination, proves to be untenable.[11] Some meanings are readily grasped by pupils without much discussion. In other cases, concept-formation is a complex process which involves both analysis and synthesis, as well as reconstruction, reorganization, discovery, and creativeness. And this is not a mechanical affair. The building of concepts demands real thinking and much experience on the part of the pupil, under the intelligent guidance of the teacher.[12]

4. The Teaching of Skills

There is no doubt whatever in the ranks of experienced teachers of mathematics that the basic mathematical skills must be taught thoroughly and systematically. And yet there is conclusive evidence that we are far from achieving even moderate success in the mastery of these skills, in spite of "the excessive emphasis commonly placed on manipulation," as one of our authoritative reports expresses it. Thus, the war years revealed a degree of all-around inefficiency in arithmetic that was truly alarming, in view of more than eight hundred hours of pupil-exposure to some work in mathematics in Grades 1–8. The convenient explanation that this inefficiency was due to "forgetting" is hardly tenable in the light of recent studies. One such study is that of Orleans and Saxe. They gave a series of five arithmetic tests, covering basic numerical skills, meanings, and applications, to some 2200 pupils in Grades 9 or 10 of 18 high schools, 11 of the schools being located in New York City. From the "Conclusions and Implications" presented in Chapter IX of their published findings, we quote these revealing statements:

> Whatever the accomplishments of the pupils might have been in arithmetic at the time they learned it, the residuum of such learning, a short time after the formal teaching of the subject has ceased, is a pitifully meager quantity. . . . *Accomplishment is poor not only as an average measure for all the pupils tested but even for the brighter groups. Even they fall far short of a reasonable standard of accomplishment.*
>
> The one general conclusion that all the findings clearly point to is that the learning has been largely if not entirely rote, with the consequent implication that the type of teaching employed led to rote learning. This conclusion is strongly supported by the data on the third test which reveal serious inadequacies in knowledge of basic items of information and the possession of basic concepts. . . . The major purpose of having pupils learn arithmetic is to enable them to solve problems. There is little evidence of problem solving as a procedure involving a reasoned technique.
>
> If the data on Test 3 serve no other purpose, they add to the evidence that much of the learning of arithmetic must have been rote, and therefore must lack significance. *It is based on lack of understanding.*
>
> Learning arithmetic by devoting 40 minutes a day every school day for eight years to addition, subtraction, multiplication, and division first with integers and then with common fractions, and so on—with all the nuances of "with carrying" and "without carrying" and cases 1, 2, 3, and 4 in adding fractions, and the several cases with per cent, and so on—is a far cry from the outcomes that are normally regarded as sound aims of learning arithmetic. THE PUPILS SIMPLY DON'T LEARN ARITHMETIC THIS WAY.[13]

Precisely similar conclusions could be reported in the field of algebra and of geometry. Years ago Professor Thorndike gave a simple algebra test in ten high

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by McLellan and Dewey (*The Psychology of Numbers*, 1895, pp. 32 ff.); by Judd, C. H. (*The Psychological Analysis of the Fundamentals of Arithmetic*, 1927, pp. 4, 79, 108).

[11] Bode, Boyd H., *Conflicting Psychologies of Learning*, D. C. Heath and Company, 1929, pp. 276 ff.

[12] Helpful discussions pertaining to the formation of concepts are to be found in such textbooks on psychology as those of Gates, A. I. (and others), of Hartmann, George W., and of Wheeler, Raymond H. (See preceding references.)

[13] Orleans, Jacob S. and Saxe, Emanuel, *An Analysis of the Arithmetic Knowledge of High School Pupils*, City College (New York) Research Studies in Education, No. 2, 1943, pp. 117 ff.

schools rated as "superior." The pupils tested had studied algebra for at least one year. From the report on that test we quote these significant comments:

> It does not seem an exaggeration to say that, on the whole, these students of algebra had *mastery of nothing whatsoever. . . . There was literally nothing in the test that they could do with anything like 100% efficiency.* . . . It would be better if they thoroughly knew what is required to master half or two-thirds of this work rather than knew a little about it all.[14]

If Professor Thorndike's test were repeated today, it is practically certain that his findings would be the same.

All of which shows that there must be something radically wrong about the teaching of the basic mathematical skills. This is certainly the case. There is overwhelming evidence that the teaching of mathematics is still conducted almost exclusively by means of repetitive drill. It appears to be the only methodology known to the average teacher. But "muscular" mathematics eventually defeats itself. It simply will not work. The reasons have been pointed out again and again.[15] And until that situation is corrected, there is little hope for mathematics.

The fatal weakness of the drill theory is that it ignores the all-important factor of "meaning" and that it confuses mechanical reaction with thinking. When the advocates of the connectionist plan of studying arithmetic calmly propose that some 1600 number facts and more than 600 unit skills be mastered by repetitive drill, they virtually "predetermine a child to failure."[16] And that is true, *a fortiori*, in algebra and geometry!

How, then, should mathematical skills be taught?

Some recent treatises on educational psychology contain very helpful chapters on the complex learning process with which we have to deal in the teaching of mathematics. The mastery of any major skill is not accomplished in a day. It demands an adequate period of growth, of maturation. *Learning a skill is a developmental process.* That is what we overlook so often in our crowded schedules. The factors of *motivation*, *insight*, and *purposeful application* are of absolutely crucial importance. Moreover, the basic skills must be *organically interwoven.* Unit skills, being mere fragments, do not coalesce without continuous effort at creating a *coherent system.* In short, there is no simple, magic formula for the effective learning of skills.[17] But an almost incredible improvement has resulted in many a classroom as soon as the teacher acquired a better understanding of the considerations which underlie the use of efficient teaching techniques.

5. Generalization—The Master Tool

A genuine mastery of the very considerable body of mathematical facts and skills would be a hopeless task if we were not endowed with the truly amazing ability which enables us to *generalize.* In fact, it is hardly an exaggeration to describe generalization as the *master tool of mathematics.* From the first steps in arithmetic to the most advanced phases of mathematical research, we must constantly depend on generalization. The primary number facts and the fundamental processes are held together and made intelligible by the basic laws of order, of grouping, and of distribution. Every formula is a generalization applying to all specific instances of the same type. A geometric theorem enunciates a relation that exists in all configurations of a certain kind. Thus, the Pythagorean

[14] Thorndike, Edward L. (and others), *The Psychology of Algebra*, The Macmillan Company, 1926, pp. 320 ff.

[15] See, especially, Brownell, William A., "Psychological Considerations in the Learning and the Teaching of Arithmetic," in *The Tenth Yearbook of the National Council of Teachers of Mathematics*, pp. 2–12.

[16] *Ibid.*, p. 6.

[17] A recent, valuable discussion of the learning process was offered by Dr. William A. Brownell, in his presidential address read before a division of The American Psychological Association, at Detroit, September 9, 1947. See, "Learning Theory and Educational Research," in *Journal of Educational Research*, March, 1948.

relation holds true of every right triangle.

In one of his most mature books, Professor Dewey comments as follows on the relation of concepts to generalization:

> Concepts enable us to generalize, to extend and carry over our understanding from one thing to another . . . . Conceptions, since they represent the whole class or set of things, *economize our intellectual efforts tremendously.* The concept signifies that a meaning has been established and remains the same in different contexts.[18]

But it was Professor Judd who never tired of stressing the indispensable role of generalization. In a well-known monograph he warns against the practice of training pupils in "relatively trivial particular skills." He goes on to say,

> *Fortunately, the mind of man is so organized that it generalizes. Even if all the curriculum-makers resolve to train nothing but particular abilities, pupils will generalize and will continue to do what the race has done throughout its history, that is, abstract from particular situations those aspects which are most universal.* Some children will acquire the general idea of mathematical exactness no matter how far curriculum-makers go in running counter to human history.
>
> All the experiments and analyses reported in this monograph lead to the conclusion that *general ideas are the most important products of instruction in arithmetic. The fundamentals of arithmetic are general ideas and general formulas, not a multitude of special skills.*[19]

That generalization does "economize our intellectual efforts tremendously," has been shown beyond any reasonable doubt, especially in the field of arithmetic. Such experiments as those of Olander and of Thiele make it certain that "a number system that contains laws or principles producing internal coherence need not be assimilated by the method of specific repetition or drill on each of its distinguishable parts."[20]

A classic demonstration of this fact was presented in an extensive report (unpublished) by Miss Irene Sauble of Detroit, at the 1940 meeting, in St. Louis, of the National Council of Teachers of Mathematics. Spectacular results were obtained when groups of second-grade pupils were given systematic training involving insight and generalization. Not only did they develop the ability to attack *new* processes with success, but they also gained a "greater sense of security in all their arithmetic work."

6. Teaching for Transfer

It is surprising that even in recent educational pronouncements one may still encounter some rather archaic remarks about the "exploded myth of mental discipline." And it is always Professor Thorndike who is given the main credit for setting off this "explosion." What Thorndike really showed was that in education, as in every other field, it is true that "as we sow, so shall we reap." By using his testing devices, he made it clear that a mere exposure to a year of Latin or Algebra does not automatically confer a magic fund of mental power to be used in any subsequent life activity. It was a good thing that such a foolish notion, wherever it existed, was rudely shattered. Common sense alone should have effectively prevented the "dogma of mental discipline," thus interpreted. However, in all this uproar about "mental discipline," educators rarely raised the prior question, as to what would happen if we had better teaching and a more adequate time allowance. Instead, they seemed to be only too glad to have another excuse for dethroning the "old-line subjects," irrespective of consequences.

Today, after decades of further research and a torrent of literature, the great tempest of "discipline" has spent its force. Long ago, it was replaced by controversies relating to the problem of a real or alleged "transfer of training." That question, too, may be regarded as practically settled. It is now agreed that "the fact of transfer cannot be doubted," but that the *amount* of genuine transfer is all a matter of

[18] Dewey, John, *How We Think*, D. C. Heath and Co., 1933, pp. 150 ff.

[19] Judd, C. H., *Psychological Analysis of the Fundamentals of Arithmetic*, University of Chicago Press, 1927, pp. 116 ff.

[20] Hartmann, George W., *Educational Psychology, op. cit.*, p. 324.

method. To quote,

> Whether transfer does or does not take place is no longer an issue. The evidence clearly demonstrates that we can carry over what we learn to novel situations. . . . The important question is what methods of teaching and learning will be most productive of transfer. . . . [Recent investigations and experiments] indicate, first of all, that *the more meaningful the learning, the more likely it is to transfer.* Rote learning, routine and blind rule-of-thumb procedures, and empty verbalism are almost certain to give disappointing results. . . . *Transfer depends upon understanding.* Second, *what always remains merely specific is not available for transfer.* . . . Finally, *transfer is not automatic.* It depends upon a deliberate attempt to interpret new situations in the light of past experience, and to apply appropriately the meanings or methods previously learned.[21]

"Teaching for transfer" is now admitted to be a central objective of education. Outstanding teachers have long acted in accordance with that position. And we may gratefully attribute to the searching debates about "discipline" and about "transfer" a definitive recognition of the all-controlling factor of *good teaching,* and of the age-old truth that "as is the teacher, so is the school."

7. Conclusion

In conclusion, this brief review has attempted to focus attention on the central problem of the school, that of skillful and effective instruction. We have emphasized the crucial need for *non-mechanistic, meaningful* teaching and learning processes, based on *motivation, insight,* and *purposeful application.* This means that great readjustments must take place in many of our mathematical classrooms.

But this necessary redirection will not come about automatically. For years, our normal schools and teachers colleges,—with some notable exceptions—, have virtually abdicated in the field of mathematics. It will take much effort to correct this unpardonable blunder which has left many of our teachers of mathematics almost totally unprepared. The present emergency situation demands the prompt application of remedial measures. All the usual programs for in-service training should be continued. However, perhaps the most helpful first-aid plan involves the organization of both regional and national institutes, financed cooperatively, which shall assist in the preparation of master teachers. Then, too, we need scores of laboratory schools and demonstration centers, operated by teachers possessing such superior training.

Finally, if we can also enlist a more active type of support on the part of our educational and administrative leaders, we may confidently hope that at last we shall succeed in our long-continued struggle to "teach mathematics into its rightful place."

[21] Gates, Arthur I. (and others), *Educational Psychology, op. cit.,* pp. 536 ff.

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◆ THE ART OF TEACHING ◆

On the Teaching of Geometry

By ROLLAND R. SMITH
Springfield, Mass.

THE aims and methods of teaching geometry have changed a good deal during the life time of many present teachers who still have the expectancy of a decade or two of service. I remember my early experience as a student of geometry in high school. I was three days late in entering school that year and the class was already on what they called Theorem V. At the end of the week I had to take a test on the first week's work. Evidently I had not understood the theorems well enough to memorize them. There were more red pencil marks on the paper when it came back to me than there had been black pencil in the first place. I told my parents I was doing well enough in everything else but geometry, but I could not make head nor tail of that subject.

The teaching was typical of the time. The teacher plunged immediately into formal demonstration without any preparatory work. It was taken for granted that we had somehow or other picked up all we needed to know about straight lines, perpendiculars, parallels, angles, triangles, and more complex figures to be able to work with them in proofs. No time was taken to become familiar with the figures themselves. No thought was given toward developing in us an understanding of proof. We were given the proofs and we memorized them. If, later, we began to understand what wewere doing, that was our good luck. The aim was quite baldly for us to remember the proofs of enough theorems to pass the college entrance examinations. And the colleges were helpful to the extent that they did not make us wait a year or two before we took the examinations. We could take the geometry examination immediately after our geometry year, get the subject off our minds, and promptly forget it.

A few years later I began the study of education preparatory to teaching. When the instructor in charge of my first course found out that I was interested in mathematics, he promptly gave me the task of writing a long paper on *Formal Discipline*. I was supposed to learn that the teaching of mathematics to discipline the mind was futile because there was no such thing as disciplinary values. Evidently I did not believe all I learned, or I was too young to appreciate it, because in due course I was teaching geometry by the time honored method, a theorem a day, take it or leave it. In the first week of geometry I even showed my pupils how to prove in a very formal step-by-step procedure that *All right angles are equal* and *When one straight line meets another the adjacent angles are supplementary.*

I soon found, however, that my pupils did not understand what they were doing. As far as meeting disciplinary aims is concerned, they might as well have tried memorizing the New York City telephone directory. I found too that other teachers were feeling as I did about the teaching of geometry. Improvement in the teaching of the subject became a favorite topic in section meetings. Soon the whole field was given a boost by nation-wide discussion of the preliminary reports of the National Committee. The National Council of Teachers of Mathematics was established at about the same time and took over

the publication of THE MATHEMATICS TEACHER. The yearbooks began to appear. The newer textbooks were written more for the pupil than for the teacher. I have seen geometry emerge from a cut and dried subject to one where a great deal of thought is given to pupils' understanding and grasp of larger aims than learning theorems. Pupils are carefully prepared to deal with the figures of geometry. The underlying concepts of demonstration are developed before formal proofs are begun. In many instances the thinking of geometry is applied not only to geometry itself but to everyday living conditions. The aims are large and the teaching has had to change to meet the aims.

Most teachers of mathematics never could subscribe to the extreme view against disciplinary values in geometry. They had seen too many of their students, particularly the better ones, seemingly carry over abilities from geometry to other fields. When we were told that there was no transfer of training except in the case of identical elements, we were wont to see a good many identical elements which may not have existed. Later when the psychologists discovered that some transfer was possible if we taught definitely for it, we tried very hard to lift our teaching out of the day-by-day theorem learning and taught not only geometry but sought to help the transfer by dealing with a similar general situation in non-mathematical subjects. The long controversy over transfer of training was good for the teaching of mathematics for, in spite of the fact that teachers were at first quite smug in their conservatism, they inevitably began to be critical of their own teaching. We no longer think of transfer as automatic. We understand much better how students learn. And now that the fact that learning gained in one field can be transferred to other fields is not only granted but hailed as a boon to education, we strive for the things that make transfer possible—general concepts, meanings, understandings—all three.

There are two aspects to the study of demonstrative geometry. There are, of course, the geometric concepts and geometric statements of relationship. The fundamental ideas are important to all the students taking the subject. Many of the details are prerequisites for students going on in mathematics or other sciences. This aspect goes without mentioning. The study of these things is geometry for geometry's sake. On the other hand, there is the demonstrative side of geometry. In this aspect of the subject lies the value of geometry for general education.

The study of demonstrative geometry should give students an opportunity to learn the meaning of deductive proof in mathematics. Logically the subject begins with axioms, postulates, undefined terms, and defined terms; then it proceeds by deduction to a sequence of theorems with exercises applying those theorems. In the process of studying the proofs of the theorems and especially in the working of abstract and practical exercises that require application of the theorems, students can gain experience with deductive proof. They can learn to distinguish between hypothesis and conclusion and can see the relation of the hypothesis to the ensuing deductive proof. They can understand the significance of the "If-Then" relationship. They can learn the futility of attempting to use any statement as an authority before the conditions of that statement are fulfilled. In fact, they usually have to ascertain first what it means to have a condition fulfilled and this can be done with a minimum of effort in the geometry classroom. In all this we have said *can* purposely because the learning does not come automatically merely by proving one thing after another. Insight must be developed by seeking zealously for meaning and bringing generalizations to the level of consciousness. According to present-day thinking, students who understand the meaning of proof, who are able to formulate proofs, and who are

capable of making intelligent criticisms of alleged proofs in geometry can use much of this knowledge in other fields.

If these larger aims are what we expect our students to get from the study of demonstrative geometry, it behooves us to analyze the learning of our students so that we can better prepare them to meet these objectives. We cannot say too often that merely going through the motions of teaching the proofs of geometry is of doubtful value. What we teachers present in the early weeks of demonstrative geometry becomes of primary importance. Can your students see the diagrams of geometry analytically? When they see, "If this is so, then that is so," do they get the logical significance of the statement? Do the proofs they study mean something to them, or do they show by their sometimes ludicrous mistakes that they are doing their best without much success to memorize what they have seen? The way teachers nowadays attempt to develop these three phases suggested by these questions is one of the main differences between contemporary teaching of geometry and the teaching of thirty years ago.

Because we believe that the study of demonstrative geometry can make a large contribution toward the general education of students who have the ability to profit by it, we think in terms of larger units than formerly at the beginning of geometry. To be sure, we deal with straight lines, angles, supplementary and complementary angles, congruent triangles, and the like, but we fit our teaching of these things into larger categories. And by so doing the details of the topics we have mentioned become clearer. To be more specific we can discuss briefly three of these larger objectives—becoming familiar with figures so that students can use them in proofs, understanding the "If-Then" relationship, and developing the meaning of deduction.

We have learned that students generally do not see the diagrams of geometry analytically. An isosceles triangle with the vertex angle bisected may appear merely as a triangle with a line through it; the two triangles within the isosceles triangle. may not be seen until you mention them. Identification of the corresponding sides and angles in overlapping triangles may be difficult. Vertical angles are quite apparent when only two lines are involved, but the addition of a few more lines may obscure the vertical angles. Alternate interior angles may be recognized in the figure as it is usually given with the theorem, but the alternate interior angles made by a diagonal in a parallelogram may appear as something quite different. Even supplementary adjacent angles cannot always be chosen correctly if there are a few extra lines in the figure. Students first see figures as wholes and must be helped to see the parts essential to a particular discussion and to dissociate these parts from the irrelevant. The smaller objective is to learn the names of the various figures and to be able to recognize them. The larger objective, is to develop within one's self the ability to take any one of these figures and see in it whatever relationship exists in the solution of a particular problem. The larger objective can be seen working through all the details. It is a *whole* that holds the details together.

Another large objective toward which much of the work of the first few weeks can be pointed is a development of the logical significance of the "If-Then" relationship. It is sad but true that we have often tried to prove a theorem like "If two sides of a triangle are equal, the angles opposite those sides are equal," for students who had no conception of the meaning of given conditions and consequences therefrom. To be sure, if you said, "If you do good work in geometry, you will get an A," students could understand that. They could see a causal relationship; the good work brought about the A. They could see a temporal relationship: the good work came before the A. But when they see an isosceles triangle, they see two equal angles and two equal sides at the same time.

Many students have told me they could see no difference (before the idea had been developed) between the two statements, (1) "If two sides of a triangle are equal, the angles opposite those sides are equal, and (2) If two angles of a triangle are equal, the sides opposite those angles are equal. Instead of starting off with the proof of theorems which are stated mostly in the if-then form, we have learned that it is a good plan to prepare carefully for the significance of such statements.

One method of dealing with this concept which seems to defy mere explanation in words is to develop it through constructions. To take only one case. Have the students construct a triangle by making two sides equal and then measure the angles. They will see that the equal sides came first and then the equal angles. In one sense they can feel that the equal sides caused the equal angles. Then deal with the converse statement in a similar way. If now you associate the "If-Then" sentence with the idea that is developing, students will begin to see what is meant. One illustration is not enough for the students. The objective will carry through several days, sometimes a few weeks. It can be applied in many construction problems and will be the thread that holds together and makes interesting many details which might otherwise be isolated.

These objectives are of course subsidiary to the largest objective of all—developing the ability to understand, make, and criticize deductive proofs. Realizing such an objective takes time and it takes careful preparation. Once again we must say that students do not learn what a deductive proof is automatically by merely going through proof after proof day after day. When we jump into proofs without preparation we are in effect assuming that our students already understand the very thing we are trying to develop. We can direct the thinking of our classes better if we keep in mind that for our students formal deduction is really new. Up to now they have been able to draw their conclusions rather informally from their total acquired knowledge. Now they must hold definitely to what is given. They are restricted as to the reasons they may use. And they must be sure that all the conditions of each reason are fulfilled before they use it. It takes time to unfold these rather strict rules of deductive reasoning. How to do it is beyond the scope of this short article. Experienced teachers are well aware of this aim and new teachers will find the literature on the teaching of geometry full of discussion of it. What we want to say here is that the teacher who keeps this aim uppermost is a different kind of teacher from the one who tries merely to show students how to prove this or that theorem, this or that exercise. The larger objective gives perspective to the whole year's work.

I have dwelt on these aims not only because of their value in orienting teaching at the beginning of geometry but because I believe we find in these more general objectives a stimulus to our thinking about the future of demonstrative geometry. We shall always, we must always, teach the facts about such things as parallel lines, parallelograms, arcs and angles, similar triangles, areas, circles, and the like. These are geometric facts that are quite necessary for many of our students. But we must remember that for every student taking demonstrative geometry for mathematics' sake there are many others who will get more value from the subject if it is taught from the point of view of general education. The Second Report of the Commission on Postwar Plans suggests that we should think of our geometry year in terms of larger units or objectives, And we can do just that without a great deal of change in our thinking. We have to raise our sights a bit.

Such topics as have been mentioned in the preceding paragraph can be studied not only from the mathematical point of view but from the organization point of view. "Organizing your knowledge" would be an objective going through the whole year.

And this kind of thing can be transferred to other fields. Another unit could be this, "The comparison of inductive and deductive reasoning." Theorems about the sum of the angles of a triangle or polygon and the relationships between arcs and angles are made to order for such an objective. In this unit would come the discussion of algebra as a means of proof for the use of letters raises the situation from a single case to all cases. A third large objective might well be "The advantages and disadvantages of deduction as compared to measurement." The geometry associated with this could be similar triangles, areas, and the measurement of polygons and circles, for the relationships here can be treated both from the deductive and the measurement point of view.

I am sure that geometry would be more interesting if its teaching were thought of in terms of these larger educational aims. For one thing, they would be small in number. Both the teacher and the student would not have to deal with so many isolated details as when working with the purely mathematical objectives. It is well established that details are better remembered when they are fitted into a purposeful whole. Some topics not so valuable toward reaching the goals could be dropped to give more time in a year that is now pretty well crowded. Teachers could finally get away from the feeling that every little sentence in a geometry book is of divine origin and must be treated without partiality. Larger objectives would give more meaning to our teaching.

And what shall we do in geometry with our "second track" students? There is so far little agreement, as a matter of fact little experience. The general mathematics of the ninth grade is fairly well established. The geometry is largely constructive, intuitive, numerical. During the next year much of this must be repeated in order to establish it and make it more functional. But I, for one, believe that sometime during the year when the kind of geometry the students have been doing is being organized into groups of interrelated facts, the students can be shown that some things can be found out without measuring. They certainly can see that if they measure one of two supplementary adjacent angles, they can compute the other angle. And when they have agreed by measurement and inductive methods that alternate interior angles with parallel lines are always equal, they can see there is no need to measure the angles of triangles to find their sum. They can be shown, quite informally if you wish, that there is a connection. And so with the second track students we can show on a less formal level what it means to make a deductive proof.

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Teaching Mathematics as a System of Understandings

By E. R. Breslich
The University of Chicago, Chicago Ill.

Competence has always been a major objective of the teaching of mathematics, but the method by which the pupil is to attain it has varied. In arithmetic the teachers have tried such methods as drill, practical applications, social utility, life situations, incidental teaching, and recently the meaning theory. Each was highly endorsed by educators and school administrators. Each enjoyed a period of popularity in the schools, only to be abandoned when its advocates had to admit that it was inadequate.

Methods of teaching arithmetic have usually influenced the teaching of high school mathematics, especially algebra, because both subjects deal with numbers. Having established a foothold in algebra it is natural that they be tried in geometry and trigonometry. The drill theory is a good illustration. Starting from the assumption that acquisition of skills is the major objective of a tool subject like arithmetic, the teacher tries to develop those skills by what seems to be the quickest and easiest technique, namely by frequent repetitions. The pupil is "drilled" until his responses to arithmetical exercises are rapid and accurate. Tests for measuring achievement are easily constructed. They are supposed to supply the evidence that "mastery" has been attained. After a long period of popularity, however, even its most enthusiastic supporters had to admit that memorizing is not learning and that drill alone is not conducive to learning. The drill method is now being replaced by the *meaning theory*, which has gained prominence steadily during the last ten years.

Since the turn of the century extensive changes have taken place in the teaching of high school mathematics. Often, however, progress has been retarded because methods which found their way into the teaching of arithmetic managed also to invade the field of secondary school mathematics. Thus having decided to use the drill method the teacher often failed because he did not realize that he must develop not only skills but also insight and understanding; that it takes more than drill to accomplish that; that he cannot teach by drill although it is useful in fixing retention of what has been taught; and that drill as a step in the learning process must come after understandings have been developed, not before. Moreover, when a person needs to use high school mathematics, the situation seldom calls for speed of performance. The important thing is that he understands what he is doing and that at every step he can verify the correctness of his work. A method of teaching which accomplishes that is far more difficult to master than the drill method. Its adoption in the schools will be slow.

The *meaning theory* in arithmetic lays emphasis on understandings. Those who advocate it speak of it as a method of teaching arithmetic as "a system of understandings of concepts, principles, processes and methods of thinking." Such a method is no task for the "lesson hearer" and for the untrained teacher. It calls for real teaching skill.

It should be kept in mind that while the meaning theory aims to develop a system of understandings it need not abandon those features of other methods that are also effective in the development of meanings but it should avoid the mistakes of these methods.

Emphasis on Understandings in High School Mathematics

In the teaching of high school mathematics emphasis on meanings and understandings is not as recent a proposal as in arithmetic. The idea was approached in the recommendations of a number of leaders in the field of mathematics at the turn of the century. The recommendations were the beginnings of important movements aiming to improve the teaching of mathematics. They emphasized understandings of basic concepts, of general principles and of processes, and methods of reasoning. The method was later endorsed by the National Committee of 1923. "The primary purpose of the teaching of mathematics," says this committee, "is to develop those powers of understanding and of analyzing relations of quantity or space which are necessary to an insight into and control over our environment . . . ." Among the aims of the teaching of mathematics the committee list: "A progressive increase of understanding of the nature of the fundamental operations . . . "; "an understanding of the language of algebra; the ability to understand and interpret correctly graphic representations; development of the ability to think clearly in terms of the ideas and concepts with which the quantitative thinking of the world is done." All of this sounds very much like the meaning theory.

One may ask what results these recommendations of the first quarter of the century have secured during the second quarter. That they have not failed to make a lasting impression on those who take leadership in the field is apparent from the reports of the later committees, the yearbooks of the National Council of Teachers of Mathematics and many of the articles that have appeared in the mathematical journals. The recommendations have been repeatedly emphasized in numerous studies. For example several studies on objectives which were published in the late twenties list development of "real understandings of the meanings of a definite number of concepts, processes, principles, and relationships" among the major objectives.

Thus for almost fifty years the development of these understandings has been recommended to the teachers of high school mathematics. However, people who have the opportunity to visit classes and who observe and analyze the errors and difficulties of pupils find that these objectives are not being attained by most pupils.

To put into actual practice the recommendations of individuals and committees is indeed a slow process. It almost seems that the meager results do not justify the large amount of time and effort spent in the preparation of the reports. To be sure after they are published they are widely read and discussed in teachers' conventions and in education classes. They are endorsed by educators and school administrators but after that interest fades. Soon they are forgotten by those for whose benefit they were written, the teachers. It should be expected that a theory which admittedly is sound and which began with the recommendations of leaders in the teaching of mathematics as far back as fifty years should have reached nationwide adoption in that time. Apparently since that has not happened there is need for a type of planning that will anticipate and remove obstacles which retard progress. If it is apparent that the teachers will encounter difficulty in putting the recommendations into practice the reports might be supplemented with or followed by pamphlets of explanation showing in detail how this may be done. The plan should provide suggestions for speeding the insertion into courses of study and textbooks. It should cultivate a sympathetic attitude among school administrators. Without such planning any future report will continue to meet the fate of its predecessors.

The following pages offer some typical examples showing how mathematics may be taught as a system of understandings

of concepts, principles, and processes and methods of thinking. They do not propose a new teaching procedure. Rather they draw upon any method which seems suitable for aiding understandings. Thus they aim to avoid the mistake of advocating the extreme use of a particular method.

Especially helpful are methods which offer pupils direct experiences such as the following.

1. To lead the pupil gradually from concrete situations to abstractions. The term *concrete* signifies anything that is understood. For example, if the pupil understands the process of adding fractions in arithmetic this may be used to help him understand the corresponding process in algebra.

2. The laboratory method which leads the pupil into discoveries of his own.

3. The experimental method by which he passes from specific experiences to generalizations.

4. Practical applications which clarify understandings and have an appeal to the pupil because they show the usefulness of what he is studying.

5. Life situations which fall within his experiences and which he can understand.

6. Social uses which show the place of mathematics in the life of the adult and of the community.

7. Applications in other school subjects.

Understanding of Concepts

The concepts of mathematics are the basis for future work. Care must be taken to make sure that they are understood. Thus meanings of such concepts as *angle*, *ratio*, *circle*, *congruency*, *similarity*, and others in geometry are important. Likewise in algebra the pupil must know the meanings of *literal number*, *signed number*, *exponent*, *formula*, and *equation*. In trigonometry he will have difficulty unless he understands the meanings of *trigonometric ratio*, *identity*, *equation*, and *logarithms*.

Two ways are commonly used to acquaint the pupil with a new concept. One starts with the definition and is followed with illustrative examples to clarify the meaning. The other provides the experiences first and lets the meanings grow out of them. The first is the method of the adult. It comes to the meaning quickly but when it is used with pupils it may not amount to more than memorization of meaningless statements. The second is the method of the learner. Gradually he is made acquainted with the characteristics and properties of the new concept. Having acquired a clear notion he learns its name and is able to give a description in his own words. Then the book or the teacher gives the definition which formulates his experiences in correct scientific language. Although the advantages of this method are generally conceded few textbooks use it consistently. The reason is that teachers fear that progress is too slow. The tendency is to get to the formal work as quickly as possible, with the result that the pupil later runs into difficulties due to his lack of understanding. Nothing is gained when progress is too rapid for understanding.

The second method is illustrated with the three examples below.

1. *Similarity of Triangles.* It will be assumed that previously the pupil has acquired the notion that congruence means figures having the same size and shape; or having the corresponding angles equal and the corresponding sides equal; furthermore he has learned that "ratio" of two line segments means that the length of each segment has been found by measurement and that one length has been divided by the other.

In developing the concept *similarity* the laboratory method may be used. First the pupil draws two triangles having two pairs of corresponding angles equal and having unequal corresponding sides. He is acquainted with the fact that the third angles are also equal. He observes that the triangles he has drawn are of the same shape, and the term "similar triangles" is introduced.

Next he finds the ratio of the corre-

sponding sides and observes that they are equal.

Now he is ready for the definition of similar triangles and polygons, which says that two triangles are similar if the corresponding angles are equal and if the ratios of the corresponding scales are equal.

2. *Exponents.* A number of investigations has shown that many errors in algebra are caused by a lack of understanding of the concept "exponent." Statements like $a^2 \cdot a^3 = a^6$, $a^2 + a^3 = a^5$, $a^2\ b^3 = a\ b^5$ illustrate some of these errors. The meaning of exponent must therefore be derived with great care. This may be done by means of concrete situations taken from geometry as follows.

The pupil finds that the area of a square drawn on squared paper is obtained by multiplying the length of a side by itself. Algebraically stated the area is $a \times a$.

The product $a \times a$ is called "a-square," the name being derived from the square whose surface has been measured.

Briefly *a-square* is written a^2. The number 2 is called *exponent* and indicates two equal factors of a product.

From a drawing or model of a cube he finds next that the volume is $a \times a \times a$. This is called *a-cube* and is written a^3, the number 3 being the *exponent* which indicates three equal factors.

Likewise, $a \times a \times a \times a$ is called *a*-fourth written a^4.

The numbers 2, 3, 4 in a^2 a^3, a^4 are called *exponents*. They indicate the number of equal factors in the products.

Although the meaning of exponents has been established for whole numbers no complete definition has been formulated. Later, as need arises the meaning of exponents will be extended to include not only whole numbers but also zero, negative numbers, and fractions.

3. *The tangent ratio.* One way to develop the meaning of this concept is to start with a practical problem: "A surveyor wishes to determine the height of a smoke stack."

He may solve this problem as follows:

a) From the foot of the stack he lays off a distance of 20 feet.

b) He then measures the angle of elevation and finds it to be 60°. A triangle is thus formed in which the base is 20 feet; the two angles adjacent to it are 90° and 60°; and the stack is the unknown side h.

c) He draws an auxiliary triangle, two angles of which are 90° and 60° while the base is of any convenient length. This triangle, he knows, is similar to the triangle of which the stack h is a side.

d) From the auxiliary triangle he computes the ratio, r, of the side opposite the 60° angle to the side adjacent. This ratio is equal to the ratio $h/20$. Hence, $h/20 = r$.

e) Knowing the value of r he solves the equation $h/20 = r$ for h.

The surveyor's work is greatly simplified by a table in which he can find the value of r for any given number of degrees. This makes steps c) and d) unnecessary.

Thus, after step b) he finds from the table that the ratio corresponding to an angle of 60° is 1.73.

Hence the equation he must solve will be $h/20 = 1.73$.

The ratio of the side opposite an acute angle of a right triangle to the side adjacent to it is called the *tangent ratio.*

Understanding of Principles

One of the mistakes of the drill theory in arithmetic has been the attempt to "teach" by drill, rather than by developping first the understandings. The desire for speed to save time, the lack of patience in gathering sufficient evidence that understandings have been attained, and the neglect of the repeated uses of the principles taught are responsible for much of the confusion in the minds of the pupils trying to learn arithmetic.

For example, in teaching how to multiply common fractions most teachers rationalize the process by using concrete materials. Next they show how the resulting product should be changed to the simplest form by using the principle that

the value of a fraction is not changed if numerator and denominator are divided by the same factor. However when the time comes for practice they do not emphasize this principle. Instead they introduce a trick, or shortcut which gives the correct result more quickly by "canceling" a factor in the numerator of one of the fractions and in the denominator of the other. In time cancellation becomes a purely mechanical performance and the principle on which it is based is soon forgotten. Later, when the pupil comes to a set of exercises which contains a mixture of multiplication and addition problems, he cancels as before, but finds that the answers in the addition problems are wrong. He becomes confused and loses the assurance with which he formerly operated.

Unless the teacher of high school mathematics insists that the work is constantly based upon broad principles his pupils will run into the same difficulty. Unfortunately too many teachers fail here. They let the operations with fractions, exponents, signed numbers, equations, etc., be based upon such mechanical processes as cancellation, transposition, clearing of fractions, removal of parentheses, etc. In fact these processes take the place of the real processes of operations, namely addition, subtraction, multiplication, division, raising to powers, and extracting roots.

Shortcuts may at first save time in getting answers, but this saving is small when compared with the large amount of time which the teacher must spend later if he is to extract the pupils from the confusion in which they find themselves.

High school mathematics must be taught as a system of laws and principles which cannot be violated. The pupil whose work is based merely on shortcuts and mechanical procedures works only with the assurance which comes from what the teacher and textbook say. The pupil whose work is based on frequently repeated principles which he understands, ultimately arrives at those forms of thinking that are characteristic of the trained student of mathematics.

Understanding of Processes

Mathematics is a system of abstract processes. The pupil who in arithmetic has acquired the habit of relying upon rules of procedure that are presented in the textbook or by the teacher, without training himself to gain real understanding of rules and processes, usually continues to memorize the rules in algebra and the demonstrations in geometry. He will fail because the processes of algebra are far more intricate than those of arithmetic. Unless teaching is such as to change these harmful methods cultivated by years of study of arithmetic he will not progress very far in high school mathematics.

The manipulative processes of algebra cannot be reduced to a routine procedure. They must be understood by the pupil. Thus, in the effort of solving a problem which requires the performance of fundamental processes of algebra the pupil must be able to satisfy the following requirements. He must

a) Be able to read and grasp the meaning of the symbols in the exercise.

b) Recognize the particular operation he is to perform first.

c) Know the appropriate procedure for this operation.

d) Take certain steps in a fixed order.

e) Know how to determine the algebraic sign of the results.

f) Select and perform the proper combination of numerical factors.

Moreover each step is likely to be itself a complicated process. Thus if he decides that a particular exercise calls for multiplication he must recall the procedure and principles involved in multiplying. In determining the algebraic sign of a product or sum he must recognize the signs of the numbers to be multiplied or added and know the procedure to be followed by one of the laws of signs. Moreover, his former experiences must be such

that he has assurance that the law is valid, that is the law must not be merely memorized but must be understood.

Before two fractions can be added the form must be changed so that the denominators are the same. When a fraction is to be changed to a simpler form it is necessary for him to know how to factor the numerator and denominator. When he has to divide by a fraction he must invert the divisor and change the operation to multiplication which itself is a process he must understand.

Without clear understandings he will commit certain errors because of partial similarities in wholly different situations. For example, the rule for multiplying two common fractions may suggest that he try to add the two fractions by adding numerator to numerator and denominator to denominator.

Because he found that $a^2\ b^2$ is the right answer to $(ab)^2$ he may submit a^2+b^2 as the right answer for $(a+b)^2$, log $a+$ log b for log $(a+b)$, and sin $a+$ sin b for sin $(a+b)$. It must be evident that unless the teacher develops an understanding of the processes the pupil cannot proceed far in the study of high school mathematics.

Understanding of Methods of Thinking

Much has been said and written on the value of mathematics as a means of training pupils in methods of thinking. No longer do teachers believe that the power of exact and critical thinking grows automatically out of the study of mathematics. If properly taught, however, the subject can make a real contribution to the development of this power.

In all mathematical subjects the processes of sound reasoning and clear thinking are easily exhibited. Thus, the ideas of *dependence*, *correspondence*, *changes*, and *relationships* are encountered everywhere in mathematics. Functional thinking is concerned with these ideas. The same ideas enter in numerous life situations. The pupil who has learned to use the methods of functional thinking could therefore be trained to apply its technique to social economic problems encountered by adults in the affairs of every day life.

Furthermore, all mathematical subjects offer opportunities for training in methods of exact reasoning. In demonstrative geometry the pupil must test the validity of every conclusion he draws. The same type of reasoning can receive attention in algebra, as in the verification of each step in the solution of an equation, and in trigonometry as in the proofs of identities. Training in exact reasoning should make use of non-mathematical as well as of mathematical situations.

A major objective of demonstrative geometry is to develop the pupils' understanding of the nature of proof, and to train him to make use of this method of thinking in whatever practical or social situations it may be applied. The entire *Thirteenth Yearbook* of the National Council of Teachers of Mathematics deals with the problem of developing the understanding of the nature of proof.

Not the least important methods of thinking are those employed in problem solving. They are concerned with the correct formulation of the problem, the methods of solving the problem, collecting and making records of data related to the problem, interpretation of the results, and checking the correctness of the solution or solutions.

Failure to train pupils in the foregoing and other methods of thinking would deprive the teaching of mathematics of one of its most valuable contributions to education.

The discussion above has tried to show how mathematics may be taught as a system of understandings of concepts, principles, processes, and methods of thinking. These understandings do not grow automatically out of the study of subject matter. They must be planned for with care for each unit and must be kept in mind during the daily work of the class. They will facilitate the pupil's progress in mathematics and add greatly to the value of mathematics in general education.

A Technique for Giving Meaning to Elementary Mathematical Operations

By NATHAN LAZAR
Midwood High School, Brooklyn, N. Y.

VERY few words in the English language have been used so frequently and so loosely in the recent literature on the teaching of mathematics, as the words "meaning," "significance," "insight," and "understanding." It would, of course, be a distinct service to clear thinking in general and to that of mathematical pedagogy in particular to analyze the various penumbra of meanings associated with each of the above words. The purpose of this article is, however, much more modest and far more restricted: It aims to show how the full meaning of mathematical operations can be made clearer to the immature student by the simple device of asking him to answer a set of three questions to be described below.

When a student is performing a simple mathematical operation like solving an equation of the type $2x+3=17$, or "reducing" the fraction $\frac{6}{8}$, he is usually asked to explain what he is doing. There are only a few teachers of mathematics left— it is hoped—who would accept the statements "transpose the $+3$ to the other side of the equation and change its sign," and "cancel," as explanations for the above two problems. Most teachers would, however, be satisfied with answers such as the following: "Subtract $+3$ from both sides (or members) of the equation," and "divide numerator and denominator of the fraction by the same number."

While the latter answers are undoubtedly superior to the former, it is the contention of this writer that they constitute only a portion of the complete process of understanding the mathematical operation. These answers give only the *description* in correct mathematical terms of what was being done. They do not make clear to the student the *two* remaining aspects of the operation which are equally essential for a *complete* understanding of the operation. The other two aspects are:

1. The purpose of the operation.
2. The justification of the operation in terms of an appropriate law of mathematics.

In the case of the solution of the above equation, the three aspects are then as follows:

1. Description of the operation—subtract 3 from both sides of the equation.
2. The purpose of the process of subtraction is, of course, to have only the unknown and its coefficient left on one side of the equation, and a number on the other side.
3. The justification of the operation of subtraction resides in the well known law of the equation.

In the case of the operation of "reducing" the fraction, the complete explanation of the process would consist of the following:

1. Description—divide the numerator and denominator of the fraction by the same number.
2. Purpose—to make both the numerator and denominator of the fraction smaller without changing the value of the original fraction.

   The purpose of decreasing the size of both numerator and denominator is very seldom explained to the student. It depends on the fact that, in general, the smaller the components of the fraction, the more *meaningful* the fraction is, and its size can be grasped more easily. Thus $\frac{1}{2}$ is more immediately meaningful than 18/36.
3. Justification—if the numerator and

denominator of a fraction are divided by the same number (or factor) the value of the new fraction is the same as the value of the original fraction.

It should be noted that this approach attempts to do the following things:

1. Take the mystery out of mathematical operations, by making their ostensible purpose clear.
2. Make the student conscious of the need of knowing the correct mathematical terminology to be used in describing the operation he performs.
3. Emphasize the fact that in all branches of mathematics—not only in demonstrative geometry—every operation that is performed must be justified by an appropriate law of mathematics.

The three phrases—(a) description of the operation, (b) purpose of the operation, (c) justification of the operation—can perhaps be expressed more succinctly and more effectively by the following pithy questions: (a) what are you doing? (b) what for? (c) by what right?

The applicability of the above technique and analysis to the entire range of mathematical operations from the simplest to the most complex is too obvious to need any further elaboration.

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